THE STRUCTURE OF NORMED ABELIAN RINGS

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Introduction. A very prominent feature of the development of the theory of function spaces, a branch of mathematics which deservedly or not has attracted very concentrated attention since the turn of the century, is that one of the essential and defining features of a function was rapidly eliminated from attention. The property to which we refer is the ring property, more properly the multiplicative property, which demands that the multiplication of any two elements be allowed. Thus when the study of these spaces was sufficiently developed to be cast into abstract form, the basic domain of operations was not a ring but merely a group, an additive group or module, with real or complex operators. When a suitable topology is introduced into such a group it becomes a space. The spaces $\mathfrak{B}$ defined by Banach seem to merit the most attention. The topology introduced is of the simplest character: a metric, invariant under translation, and homogeneous with respect to scalar (that is, real or complex numerical) multiplication. The elements of these metric groups are called vectors.

It is only very recently that its birthright has been restored to this theory and that function spaces are being studied as rings and not merely as groups. The present address will attempt to delineate some aspects of the recent developments, to point to certain achievements, and to suggest some problems. The compilation of problems, the formulation of open questions, is usually a hazardous matter. It is frequently difficult to understand the importance of a question before it has been answered. Thus our suggestions will be largely tentative.

Definition and examples of normed rings. To begin with, a Banach space is an additive group of elements $a, b, c, f, g, \cdots$ with operators $\alpha, \beta, \lambda, \mu, \cdots$ which are complex numbers (in the present work real scalars will be excluded). Every element a has a norm $|a|$ (also written $\|a\|$ or even $|||a|||$!) which is a non-negative number. The space is metrized by the norm with dist $(a, b) = |b-a|$; it is complete in this norm. Finally the norm is homogeneous, that is, $|\alpha a| = |\alpha| \cdot |a|$. We are now in a position to define a normed abelian ring.

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DEFINITION. *A collection $\mathfrak{R}$ of elements is called a normed abelian ring if*

(i) *$\mathfrak{R}$ is a Banach space.*

(ii) *$\mathfrak{R}$ is a ring with unit*[1] *e. Multiplication satisfies the axiom $ab=ba$ (abelian character).*

(iii) *Under multiplication, the norm is subject to the inequality*

$$|a\cdot b| \leqq |a|\cdot|b|. \tag{1}$$

These axioms were first given by Nagumo in 1936.[2] To indicate the versatility of the concept here defined some examples will be cited. At the head of the list will be placed certain classic rings of analysis.

1. The ring of complex numbers.

2. The ring of complex valued functions $f(x)$ bounded and continuous on a metric space S with $|f| = \text{l.u.b.}_S |f(x)|$ (for example, S might be the interval $0 \leqq x \leqq 1$).

3. The ring of functions $f(z)$ analytic over some bounded domain D and continuous over the closure $\overline{D}$ of D with $|f| = \text{l.u.b.}_{\overline{D}} |f(z)|$.

4. The ring of complex valued functions $f(x)$ of bounded variation on an interval $a \leqq x \leqq b$ with $|f| = \text{l.u.b.}_{[a,b]} |f(x)| + V_{[a,b]} f(x)$.

5. The ring of absolutely convergent Fourier series $f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}$ with $|f| = \sum_{-\infty}^{\infty} |a_n|$.

6. The ring of operators $\phi(A)$ defined by means of functions $\phi(\lambda)$ which are measurable and bounded with respect to a resolution of the identity $E(\lambda)$, $-\infty < \lambda < \infty$, in a reflexive vector space.

7. The ring generated by any bounded linear operator T defined over an arbitrary Banach space $\mathfrak{B}$ (or ring $\mathfrak{R}$) with $|T| = \text{l.u.b.} |Tf|$, $|f| = 1$, $f \in \mathfrak{B}$. The ring then consists of the polynomials in T with their limits in the indicated topology (uniform topology).

8. The ring generated by any one-parameter group or semi-group of linear transformations in a Banach space.

9. The ring of complex valued functions $f(x)$, $a \leqq x \leqq b$, possessing m continuous derivatives with $|f| = \sum_{n=0}^{m} (n!)^{-1} \text{l.u.b.}_{[a,b]} |f^{(n)}(x)|$.

10. The ring $\mathfrak{R}$ of complex valued functions $f(x)$, $a \leqq x \leqq b$, possessing infinitely many derivatives in which the norm is introduced as follows: Let $M_n > 0$, $n = 0, 1, 2, \cdots$, be a sequence of constants with $M_n/(M_r \cdot M_{n-r}) \geqq n!/[r!(n-r)!]$, $r = 0, 1, \cdots, n$. Then $f(x) \in \mathfrak{R}$ if $\sum_{n=0}^{\infty} (M_n)^{-1} \text{l.u.b.}_{[a,b]} |f^{(n)}(x)| < \infty$. This series serves as the norm of $f(x)$.

[1] There is little point in investigating rings without unit. But if R has no unit, it can be embedded in a normed ring R_1 with unit and such that R is a maximal ideal in R_1.

[2] M. Nagumo, *Einige analytische Untersuchungen in linearen metrischen Ringen*, Jap. J. Math. vol. 13 (1936) pp. 61–80.

At this point a few bibliographical remarks seem pertinent. Example 1 is the core of every ring. It will be seen shortly how the complex numbers can be distinguished from other rings. Rings in 6 were first considered by v. Neumann and Stone for Hilbert space.[3] The author subsequently showed that all the essential phenomena could be reproduced in a Banach space providing it was reflexive.[4] Since then this particular problem has been considered by F. Riesz who founds his discussion not on normed rings but partially ordered spaces.[5] Example 7 is particularly important in the present discussion. It will be seen later in our treatment of maximal ideals that a ring 7 is locally homomorphic to rings of type 2 or 3 or 9 or 10. The ring 7 where T is a bounded self-adjoint transformation in Hilbert space has a structure which permits with ease the finding of the resolution of the identity of T (which, incidentally, is not usually in $\mathfrak{R}$). This will be shown below. Although the determination of the structure of transformations by operating with their rings (rather than with the clumsier spaces over which they are defined) has had very signal successes, there are some points at which this approach has not yet yielded ground. An ideal-theoretic procedure for determining in 8 the resolution of the identity of a one-parameter strongly continuous group U_s of unitary transformations on Hilbert space has failed up to this moment.[6] A fortiori, for the case of rings 8 generated by semi-groups in Banach spaces attempts by the author to extend the results of Hille[7] have not been successful. The rings 10 are similar to those treated in the literature.[8] Incidentally, as an example of constants M_n satisfying the given conditions one may take $M_n=(n!)^s$, $s\geqq 1$. If $s=1$ the ring contains exclusively analytic functions. The results of Gelfand relating to ring 5 are well known to analysts. However the parent theory of which these results on absolutely convergent

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[3] J. v. Neumann, *Über Funktionen von Funktionaloperatoren*, Ann. of Math. (2) vol. 32 (1931) pp. 191–226; M. H. Stone, *Linear transformations in Hilbert space*, Amer. Math. Soc. Colloquium Publications, vol. 15, 1932, chap. 6.

[4] E. R. Lorch, *On a calculus of operators in reflexive vector spaces*, Trans. Amer. Math. Soc. vol. 45 (1939) pp. 217–234.

[5] F. Riesz. *Sur quelques notions fondamentales dans la théorie générale des opérations linéaires*, Ann. of Math. (2) vol. 41 (1940) pp. 174–206.

[6] This is a theorem due to Stone to the effect that $U_s=\exp(isH)$, H self-adjoint. The fact that H may be unbounded causes the difficulty.

[7] E. Hille, *Notes on linear transformations*. II. *Analyticity of semi-groups*, Ann. of Math. (2) vol. 40 (1939) pp. 1–47. See also more recent announcements in the Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 175–178 and 421–424; vol. 30 (1944) pp. 58–60.

[8] See for instance S. Mandelbrojt, *Analytic functions and classes of infinitely differentiable functions*, The Rice Institute Pamphlet vol. 29 (1942) No. 1.

Fourier series are merely a biproduct is not so widely appreciated.[9]

Reducibility of rings. In the ring $\mathfrak{R}$, every element a gives rise to an operator T_a by virtue of the correspondence $x \to T_a x = ax$, x in $\mathfrak{R}$. The operator T_a is linear and bounded with $|T_a| \leqq |a|$. This gives a representation of $\mathfrak{R}$ as a ring of operators; it is the *regular representation*. By introducing a new norm in $\mathfrak{R}$ equivalent to the old, we have $|T_a| = |a|$. We shall assume that this has been done. This change is not significant as almost all the results which we give later are invariant under algebraic and topological isomorphisms. Thus we see that the theory of linear transformations may be applied to determine the structure of the ring. The complex numbers λ for which $(a-\lambda e)^{-1}$ does not exist constitute the *spectrum* of a. On the other hand, the properties of linear transformations may be studied in very large part in rings. For if T is a linear transformation over a space $\mathfrak{B}$, T can be embedded in a ring $\mathfrak{R}$ in which it has the same spectrum as over $\mathfrak{B}$. The union of the concepts and methods of linear transformations and rings produces a numerous and healthy progeny.

The fundamental formula for studying the reduciblity of a ring $\mathfrak{R}$ is the Cauchy integral

$$j = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta e - a}, \tag{2}$$

where a is any fixed element and C is any simple rectifiable curve lying in the complex plane and avoiding entirely the spectrum of a. It is easily shown that $j^2 = j$. If for some a and C, $j \neq e$ or 0, and only then, will $\mathfrak{R}$ be reducible. An integral of the type (2) expressed for operators and in slightly different notation was given many years ago by F. Riesz in his book.[10] Riesz mentions very briefly the elementary properties of j. The results of Riesz seem to have been forgotten in the last decade; there is no mention of them in the literature, particularly that literature which could have employed them to considerable advantage. The present author rediscovered (2) and elaborated certain of its consequences with some completeness.[11] This particular Cauchy integral is a powerful tool and may serve to shorten very considerably solutions which have recently appeared. We shall state

[9] I. Gelfand, *Normierte Ringe*, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 1–24. The applications of the abstract theory to absolutely convergent series is found in the same volume: *Über absolut konvergente trigonometrische Reihen und Integrale*, pp. 51–66.

[10] F. Riesz, *Les systèmes d'équations linéaires*, Paris, 1913, pp. 117–121.

[11] E. R. Lorch, *The spectrum of linear transformations*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 238–248.

the facts on the reducibility of R in the form of a theorem.[12]

THEOREM 1. *The ring $\mathfrak{R}$ is irreducible if and only if the spectrum of every element a in $\mathfrak{R}$ is a connected set.*

If $\mathfrak{R}$ is reducible, it is the direct sum of two rings, $\mathfrak{R}=\mathfrak{R}_1+\mathfrak{R}_2$. Furthermore $\mathfrak{R}_1$ is isomorphic to the quotient ring $\mathfrak{R}/\mathfrak{R}_2$. Thus the study of reducibility is at its most demanding level the study of ring homomorphisms. The general homomorphisms will claim our attention later.

If $\mathfrak{R}$ is irreducible it may under proper circumstances be embedded in a ring $\mathfrak{R}'$ where that property abounds. This may be essayed in case $\mathfrak{R}$ is a ring of linear transformations defined over a Banach space $\mathfrak{B}$. Then $\mathfrak{R}'$ is taken to be the ring of bounded operators over $\mathfrak{B}$ which are limits of operators in $\mathfrak{R}$ in some topology weaker than the uniform—for example, the strong or the weak topology. This procedure is that which has been used to obtain the structure of a self-adjoint transformation in Hilbert space or of a one-parameter group of unitary transformations, also in Hilbert space.

Maximal ideals. The importance of the contributions of Gelfand in the theory of normed rings arises from the fact that he first considered the ideals of such rings.[13] His results deal almost exclusively with maximal ideals. An ideal $\mathfrak{J}$ is a subset of $\mathfrak{R}$ closed in the usual algebraic sense and in the topological sense as well. A maximal ideal is one distinct from $\mathfrak{R}$ which has no proper extensions. The existence of maximal ideals may be established easily by the use of Zorn's lemma. Every ideal $\mathfrak{J}$ is embedded in one or more maximal ideals. The maximal ideals are characterized by the fact that their residue classes contain precisely one scalar λe. Thus if $x\in\mathfrak{R}$ then $x\equiv\lambda e$ $(\mathfrak{J})$ where $\mathfrak{J}$ is maximal. This gives rise to complex valued functions $x(\mathfrak{J})=\lambda$ defined over the class of maximal ideals. Gelfand introduces a topology (in fact, more than one topology is used) in the class $\mathfrak{M}$ of maximal ideals in a standard fashion so that $\mathfrak{M}$ becomes a bicompact Hausdorff space. Over this space, the functions $x(\mathfrak{J})$ are constrained to be continuous. Thus the following theorem is obtained.[14]

THEOREM 2. *The mapping $x\to x(\mathfrak{J})$ is a homomorphism which maps*

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[12] E. R. Lorch, *The theory of analytic functions in normed abelian vector rings*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 414–425; in particular, Theorem 2. The more important problems dealt with in this paper are treated in the first half of the present address.

[13] See footnote 9.

[14] *Normierte Ringe*, Theorem 10.

the ring $\mathfrak{R}$ onto a subring of the ring of functions continuous on the set $\mathfrak{M}$ of maximal ideals in $\mathfrak{R}$.

Two facts should be noted. First, the ideal $\mathfrak{K}$ which maps into the function which is identically zero is the radical of $\mathfrak{R}$ (see below). Secondly, assuming that the radical is empty or has been factored out, the situation is far from clear. The ring of functions $x(\mathfrak{J})$ is a subring of the ring of continuous functions on $\mathfrak{M}$ and in this subring the norm is not (necessarily) the norm of uniform approximation. Thus this theorem gives an initial but not definitive result on the structure of $\mathfrak{R}$. More penetrating results of this type will be considered in the second half of this address.

The operation of forming factor rings $\mathfrak{R}/\mathfrak{J}$ will be performed frequently. The topology is introduced into $\overline{\mathfrak{R}}=\mathfrak{R}/\mathfrak{J}$ as follows: If $\bar{a}\in\overline{\mathfrak{R}}$ is the image of a in $\mathfrak{R}$ then $|\bar{a}|=$g.l.b. $|a+q|$ where q ranges over $\mathfrak{J}$. With this norm $\overline{\mathfrak{R}}$ is a (complete) normed ring.

The radical. A nilpotent is an element a in $\mathfrak{R}$ such that for some positive integer n, $a^n=0$. To define the radical of $\mathfrak{R}$ it is necessary to introduce the notion of quasi-nilpotent. A *quasi-nilpotent* is an element a such that for every complex number μ, $\lim_{n\to\infty}(\mu a)^n=0$. The structure of the radical has been given by Gelfand.[15] It can equally well be determined from a lemma used by the author to determine the structure of transformations.[16] The portion of the lemma we shall need follows.

LEMMA. *Let the curve C in formula (2) be the unit circle. Then j in (2) satisfies the equation $j=\lim_{n\to\infty}(e-a^n)^{-1}$. Furthermore $jb=b$ if and only if $\lim_{n\to\infty}a^nb=0$.*

Suppose that a has only the singularity $\lambda=0$. Then the same is true for $(\mu a)^n$. The integral (2) applied to μa with C the unit circle gives $j=e$. Applying the lemma, $je=e$ implies that $\lim_{n\to\infty}(\mu a)^n=0$. Conversely, if $\lim_{n\to\infty}(\mu a)^n=0$, then for all μ, the spectrum of μa lies in the unit circle, hence consists of the one point $\lambda=0$.

THEOREM 3. *The quasi-nilpotents in $\mathfrak{R}$ are those elements a whose spectrum is the one point $\lambda=0$.*

Applying the theory of maximal ideals it is clear that the quasi-nilpotents constitute an ideal, the radical $\mathfrak{K}$ of $\mathfrak{R}$. The radical is the intersection of all maximal ideals. Obviously, the ring $\mathfrak{R}/\mathfrak{K}$ is without radical. It is this ring $\mathfrak{R}/\mathfrak{K}$ which according to the previous section is

[15] See footnote 9.

[16] See footnote 11, Theorem 7.

isomorphic to a function ring. For if x is in $\mathfrak{R}$, then $x(\mathfrak{J})=0$ for all maximal ideals $\mathfrak{J}$.

Since the spectrum of the radical is concentrated at one point all methods for cracking the spectrum into pieces, in particular, methods applicable to connected spectra, are naturally of little value. The structure of the radical is largely unknown. In fact the class of examples of radicals is as yet quite limited.

Rings which are fields. A very vital question to all of mathematics is the determination of whether there exist normed fields which are extensions of the complex field. This problem was settled by Mazur in 1938[17] by proving that no norm could be introduced in the field of rational functions with complex coefficients. Mazur also established the fact that if in a normed ring we have instead of the inequality (1) the equality $|a\cdot b|=|a|\cdot|b|$, then $\mathfrak{R}$ is the field of complex numbers. Both of these results seem to be most fittingly proved by the methods of the present theory. To establish the first, use is made of the fact that the spectrum of an element is not empty. This was proved for linear transformations by A. E. Taylor.[18] Armed with this theorem, the proof is completely trivial. The second result is obtained by an appropriate application of our lemma and some associated properties of linear transformations.[19]

Analytic functions in $\mathfrak{R}$. It was noted very early in the development of linear operations that the notion of analytic function was destined to play here a very considerable role. Probably the first example of such a function was given by the equation for the resolvent $(e-a)^{-1}=e+a+a^2+\cdots$ which is valid for $|a|<1$. An important consequence of this equation is that the *regular* elements in $\mathfrak{R}$ (that is, those elements which have an inverse—also called the nonsingular elements) form an open set. A very large literature has been created about general analytic function theory. It will be impossible to give here an account of it which is in any sense adequate. However, in order to outline the principal results of the first part of this paper it will be necessary to touch on and examine briefly some of the points of view which have been advanced. It is hoped that the brevity of our discussion will not produce a distortion of the situation.

A point of view frequently employed is that which conceives analytic functions as defined over the complex plane and having values which lie in some Banach space. Such a view does not allow one to

[17] S. Mazur, C. R. Acad. Sci. Paris vol. 207 (1938) pp. 1025–1027.

[18] A. E. Taylor, *The resolvent of a closed transformation*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 70–74.

[19] See footnote 11, Theorem 8.

realize the product of two functions. Another view considers functions once more defined for complex numbers and whose values are operators over some Banach space. Here the operation of product of analytic function is permissible but that of forming a function of a function is inadmissible. Although these and still other approaches have "generality," it is still to be decided by each individual worker in this field whether they deserve to be considered as legitimate extensions of the theory of analytic functions. Our own decision is adverse.

A solution to the problem of extending suitably the scope of analytic function theory was presented to this Society by the author slightly over two years ago. It will be described briefly. The domain over which this theory is developed is any normed abelian ring $\mathfrak{R}$. The functions $f(z)$ treated here are assumed to be defined over a region in $\mathfrak{R}$ and to have their values in $\mathfrak{R}$. It is rather remarkable that the development of the theory parallels very closely the classic course in its methods and principal identities. Naturally there are points which require considerable care in the case of rings which for the case of the complex plane are non-existent or trivial. Starting with a definition of derivative couched in the form

$$|f(z_0 + h) - f(z_0) - hf'(z_0)| < \epsilon |h| \tag{3}$$

(for $|h| < \delta$, and so on) the Cauchy theorem is rapidly obtained for regions convex in $\mathfrak{R}$ or those which are the sum of finitely many such. The general region in $\mathfrak{R}$ is of a complicated topological structure and the Cauchy theorem has not been investigated for it. What is here required is that the "interior" of a curve lie inside the region. This will be touched upon later.

In the next place the Cauchy formula is established. Once more the choice of the curve C is critical. For example the formula (2) yields not merely the values 0 and e but all other idempotents j. A whole class of complications of this sort may be avoided at the outset by assuming that $\mathfrak{R}$ is irreducible. The cycle of fundamental theorems is completed with the development of an analytic function into a power series. This series converges in the largest sphere with center at the point of development in which the function is analytic. It may converge for points outside this sphere. For example, any series converges for all points in the radical.

In the development of the theory of analytic functions up to the present point, the ideals in $\mathfrak{R}$ have played no role. There is a class of ideas in which the mixture of the two concepts is fruitful. We shall give one application of this procedure. The proof is based on the

Taylor series expansion for these functions.

THEOREM 4. *If $f(z)$ is a function analytic in a region D in $\mathfrak{R}$ and $\mathfrak{J}$ is any ideal, then $\bar{f}(z)$ is analytic in the region $\bar{D}$ of the ring $\mathfrak{R}/\mathfrak{J}$. Here $\bar{f}(z)$ and $\bar{D}$ represent the images of $f(z)$ and D under the homomorphism $\mathfrak{R}\to\mathfrak{R}/\mathfrak{J}$.*

It should be noted that, under a homomorphism, a sphere in $\mathfrak{R}$ is mapped into a sphere in $\bar{\mathfrak{R}}$ of the same radius. Thus sphere of convergence goes into a subsphere of the sphere of convergence. If the ideal $\mathfrak{J}$ of the theorem is a maximal ideal, then $\bar{f}(z)$ is a function analytic in the classic sense since $\mathfrak{R}/\mathfrak{J}$ is the field of complex numbers. Thus we see that the ring of functions analytic over a certain domain in $\mathfrak{R}$ may be mapped homomorphically on a certain ring of functions analytic in the complex plane. The set of functions analytic in a domain of $\mathfrak{R}$ is considerably extensive; it includes as a proper subset all functions analytic in the ordinary sense (yet considered as generating functions in $\mathfrak{R}$). The above theorem is merely one of a host of homomorphism theorems which may be developed in this field. If the homomorphism proceeds in the direction: function of a complex number to function in a general ring, $f(\lambda)\to f(a)$, it is commonly referred to as an operational calculus. As we shall meet more homomorphism theorems later it is worth noting the specific hypotheses which support them.

A further homomorphism may be indicated at this point since it has received attention in the literature. Suppose C is an arbitrary rectifiable arc in $\mathfrak{R}$, closed or not, $\phi(\zeta)$ is a function continuous on C and a is an element in $\mathfrak{R}$ for which $(\zeta-a)^{-1}$ exists, ζ on C. Then $(\zeta-z)^{-1}$ exists for z in a certain neighborhood of a and

$$f(z)=\frac{1}{2\pi i}\int_C \frac{\phi(\zeta)}{\zeta-z}\,d\zeta \tag{4}$$

is easily shown to be analytic in this neighborhood. If now C is taken to be a simple closed curve in the complex plane, z is held fast at the value a where a has its spectrum entirely within C and if $\phi(\zeta)$ is a function of the complex variable ζ analytic in the classic sense in a region containing C and its interior then we may say that $\phi(\zeta)$ in (4) defines the element $\phi(a)$ in $\mathfrak{R}$ (instead of $f(z)$ or $f(a)$). This gives an operational calculus $\phi(\lambda)\to\phi(a)$ which has been treated by Gelfand, Dunford, and A. E. Taylor.[20] The element $\phi(a)$ is termed an "ana-

[20] Gelfand, loc. cit. Theorem 19; N. Dunford, *Spectral theory*, Bull. Amer. Math. Soc., vol. 49 (1943) pp. 637–651, in particular p. 641; A. E. Taylor, *Analysis in complex Banach spaces*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 652–669, Theorem 9.

lytic function of a."[21] Certain important functions of a are excluded by this device. Suppose for instance that the spectrum of a is the circle $|\lambda| = 1$. Then since $\log \zeta$ is not analytic in any region containing this circle, $\log a$ is not produced by this operational calculus. However the function $\log z$ may well be defined for $z = a$. An explanation for this phenomenon lies in the fact that $\log a$ does not exist in the subring of $\mathfrak{R}$ generated by a since the spectrum of a in the smaller ring consists of the solid circle $|\lambda| \leqq 1$.

Up to the present moment there is little known of the character of the domains over which analytic functions are defined and of the Riemann surfaces associated with them. An example will indicate some of the phenomena which may be expected. The regular elements a in $\mathfrak{R}$ (those for which a^{-1} exists) form a topological group. This group consists of one or more maximal connected open components or domains.[22] If the number is more than one, it may be shown that it is infinite. The component containing e is called the principal component. The function $\log z = \int_e^z \zeta^{-1} d\zeta$ is defined precisely for the elements z in the principal component. It is periodic and has for its primitive periods the elements $2\pi ij$ where j is any idempotent. These idempotents may be finite in number in which case one is dealing with a direct sum of irreducible rings; they may be denumerable or even nondenumerable. An example of the last is furnished by rings generated by suitable resolutions of the identity in reflexive spaces. This gives one a preliminary idea of the Riemann surface for $\log z$ which may have nondenumerably many sheets. In this connection the following extension of the Poincaré-Volterra theorem may be cited. Its proof is quite simple.

THEOREM 5. *If a ring $\mathfrak{R}$ is separable, then any analytic function defined over a region in $\mathfrak{R}$ has at most denumerably many values.*

It is not clear whether the same result applies to non-separable rings which are irreducible.

It is apparent from the example $w = \log z$ just considered that a periodic analytic function may have more than two periods; indeed it may have any number, finite or infinite, of periods. But if the ring is irreducible the picture becomes different. It seems probable that all primitive periods after the first two will lie in the radical. That this situation can arise may be shown by examples.

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[21] As a special case, this terminology gives in the ring of complex numbers: sin 2 is an analytic function of 2!

[22] The first discussion of these components was given by Nagumo. See footnote 2.

For irreducible rings the notion of interiority to simple curves can be established by means of the integral (2) whose only possible values in this case are 0 and e. A simple curve $C: z=z(t)$, $0 \leqq t \leqq 1$, is one for which $t_1 \neq t_2$ implies $[z(t_1)-z(t_2)]^{-1}$ exists. The interior of C is then defined to consist of the points for which the integral equals e. It may be shown that interior and exterior points do exist. Although the interior and exterior are open sets they are not necessarily connected.

The question of ring extensions was raised in connection with reducibility. The question arises again in this form. If a is singular in $\mathfrak{R}$ does there exist an extension $\mathfrak{R}_1$ of $\mathfrak{R}$ in which a is regular?[23] Further if a is regular in $\mathfrak{R}_1$, does there exist an extension $\mathfrak{R}_2$ of $\mathfrak{R}_1$ in which a is in the principal component (of the group of regular elements)? If the answer to this is affirmative, then a may be embedded in a one-parameter group. This is a question "in the large." An alternative formulation of this problem is: Under what circumstances does $\log a$ exist in a ring extension?

Rings with one generator. Let $\mathfrak{R}$ be any normed ring and let a be arbitrary in $\mathfrak{R}$. Consider the subset of $\mathfrak{R}$ consisting of the polynomials in a (and e) and their limits. This set is a ring which is a subset of $\mathfrak{R}$ and will be designated by $\mathfrak{R}\{a\}$.[24] It is clear that the spectrum of any element in $\mathfrak{R}\{a\}$ will include the spectrum of the same element considered as belonging to $\mathfrak{R}$. The expansion and shrinkage of spectra under ring contraction and extension is subject to certain laws. In particular, spectra which are purely real are absolute.[25] From this point forward we shall treat exclusively rings $\mathfrak{R}\{a\}$.

The elements in $\mathfrak{R}\{a\}$ may properly be called functions of a. The nature of these functions will be examined. For example if $\mathfrak{R}$ is the ring of all continuous functions $f(x)$, $0 \leqq x \leqq 1$, with $|f| = \text{l.u.b.}_{[0,1]} |f(x)|$, then by the Weierstrass approximation theorem, $\mathfrak{R}=\mathfrak{R}\{x\}$. The elements of $\mathfrak{R}\{x\}$ are in this case continuous functions of x. This and the examples 3, 9, and 10 introduced earlier indicate that each ring $\mathfrak{R}\{a\}$ defines a special type of function of a; in particular although every ring $\mathfrak{R}\{a\}$ contains analytic functions of a (for example, $\exp(a)$, $\sin a$), it does not except in special cases consist exclusively of such functions. It will be seen later that rings $\mathfrak{R}\{a\}$ are locally homomorphic to rings of types 3, 10, or 9. Theorem 2 is an example of a homomorphism to a ring of continuous functions. Our

[23] A contribution to this problem has been made by G. Silov, *On the extension of maximal ideals*, C. R. (Doklady) Acad. Sci. URSS. vol. 29 (1940) pp. 83–84.

[24] Observe that in the notation current in abstract algebra, $\mathfrak{R}[x]$ and $\mathfrak{R}(x)$ indicate extensions of $\mathfrak{R}$. For us here, $\mathfrak{R}\{a\}$ is a contraction of $\mathfrak{R}$.

[25] See footnote 12, Theorem 13.

study may be considered as a reformulation and the first steps in a solution of the frequently considered *problem of determining the analytic functions of an operator*. The details of this undertaking will appear separately; in what follows, we shall give a brief review of the principal results. The problem may be recast in the following form: *To determine conditions in order that a ring* $\mathfrak{R}\{a\}$ *should be homomorphic to a function ring of a given type.* The types we shall discuss are (i) The functions analytic in a domain and continuous on its boundary; (ii) The functions infinitely differentiable on some interval. The case of functions of class C^m, $m \geqq 1$, can be elaborated on the basis of those of class C^∞. The case of functions of class C^0 is handled by Theorem 2.

The study of rings $\mathfrak{R}\{a\}$ may be considered as a study in polynomial approximation since all elements of $\mathfrak{R}\{a\}$ are limits of polynomials in a. The approximation is carried out over the spectrum of a. The norm for the approximation is at least as powerful as the uniform norm since $|p(a)| \geqq |p(\lambda)|$. The *resolvent set* of a (the set complementary to the spectrum) is always connected.

If a is a quasi-nilpotent then the ideal (a) is the only maximal ideal in $\mathfrak{R}\{a\}$ and is identical with the radical. Examples show that $\mathfrak{R}\{a\}$ may have a radical even when a is not a quasi-nilpotent.

Types of maximal ideals. We shall introduce a classification of maximal ideals in rings $\mathfrak{R}\{a\}$. This is an initial step in the study of the structure of the class of all ideals. For the rings treated here the maximal ideals are easily singled out. They are precisely the principal ideals $(a-\lambda\varepsilon)$ where λ is in the spectrum of a. Furthermore, Gelfand's topology of the class of maximal ideals coincides with the usual metric of the complex plane. If a is singular then for any $n \geqq 1$ the ideal (a^n) includes the ideal (a^{n+1}). The inclusion may or may not be a proper one. In what follows, the inequality $(a^n) \neq (a^{n+1})$ will indicate proper inclusion. Also, if for an integer n, $(a^n) = (a^{n+1})$, then for all integers $r \geqq 1$, $(a^{n+r}) = (a^{n+r+1})$. This leads us to our basic classification.

DEFINITION. *For the ideal* (a), *one of the three following cases will occur*:

(i) $(a) = (a^2) = \cdots = (a^n) = \cdots$.

(ii) $(a) \neq (a^2) \neq \cdots \neq (a^n) = (a^{n+1}) = \cdots$.

(iii) $(a) \neq (a^2) \neq \cdots \neq (a^n) \neq \cdots$.

The ideal (a) *will be termed an ecto-, meso-, or endo-ideal according as it falls under case* (i), (ii), *or* (iii).

All three cases may arise. One may see that from a consideration of rings $\mathfrak{R}\{x\}$ in examples 2, 9, and 3 respectively. A more penetrating example of (i) is found in the ring $\mathfrak{R}\{H\}$ generated by a self-adjoint transformation in Hilbert space. Here the spectrum of H is real and every ideal $(H-\lambda I)$ is an ecto-ideal.

By Theorem 2 the ring $\mathfrak{R}\{a\}$ is homomorphic to a ring of functions continuous over the spectrum of a. For ecto-ideals $(a-\lambda e)$, little more can be said. However if the ideals $(a-\lambda e)$ are meso- or endo-ideals the "functions" $f(a)$ are differentiable and may even be analytic. The precise facts will be stated below. We shall, however, give at this point a consequence of the endo-concept. Suppose that (a) is an endo-ideal and that f is in $\mathfrak{R}\{a\}$. Then for any integer n we have a unique decomposition

$$f = \beta_0 e + \beta_1 a + \cdots + \beta_n a^n + q_{n+1}, \tag{5}$$

where $q_{n+1} \in (a^{n+1})$. This equation may be regarded as a form of Taylor theorem with remainder. It asserts that there exist projections of $\mathfrak{R}\{a\}$ on the ideals (a), (a^2), $\cdots$. These projections are the bounded linear transformations T_n such that $T_n f = q_n$. Since $T_n q_n = q_n$, $T^2 = T$.

It may be shown that if λ is an interior point of the spectrum of a, then the ideal $(a-\lambda e)$ is an endo-ideal. The frontier points of the spectrum naturally call for delicate handling. Rather little is known about them at the present time.

A useful concept is that of a *normed power series ring*. Such a ring will consist of abstract power series which have been normed in some manner. In addition a ring of power series must contain all polynomials and convergence in the ring must imply the convergence of the coefficients of every power. Formula(5) asserts that the mapping $f \to \{\beta_0, \beta_1, \cdots\}$ transforms $\mathfrak{R}\{a\}$ into a ring of power series. In fact, if we set $\mathfrak{I} = \prod_{n=1}^{\infty}(a^n)$ the ring of power series is isomorphic topologically (by definition) and algebraically to the quotient ring $\mathfrak{R}\{a\}/\mathfrak{I}$.

Rings of power series are not limited to rings of analytic functions. Rings of infinitely differentiable functions, in particular quasi-analytic rings, are of this type. In addition there exist rings which are the direct sum of a radical and the field of complex numbers, that is, rings which are almost entirely radical, which are power series rings. In these such series as $\sum_{n=0}^{\infty} n! a^n$ may converge absolutely.

Rings of analytic functions. If λ is any point in the spectrum of a, $(a-\lambda e)$ is a maximal ideal. Thus if $f \in \mathfrak{R}\{a\}$, by Theorem 2, f is mapped homomorphically on a complex valued function $f(\lambda)$ defined

over the maximal ideals in $\mathfrak{R}\{a\}$, in this case, the spectrum of a. This function is continuous over the spectrum. For suppose $\{p_n(\lambda)\}$ is a sequence of polynomials such that $\lim_{n\to\infty} p_n(a)=f$. Then $p_n(a)\equiv p_n(\lambda)e \ (a-\lambda e)$; and since $|p_n(a)-p_m(a)| \geqq |p_n(\lambda)-p_m(\lambda)|$ the convergence of $p_n(\lambda)$ to $f(\lambda)$ is uniform.

Suppose now that the spectrum of a contains a neighborhood D of the origin. Then $f(\lambda)$ is not merely continuous but also analytic in this neighborhood. A preliminary result is then that $\mathfrak{R}\{a\}$ is homomorphic to a ring of functions analytic in D and continuous over the closure of D. Let us suppose that under this homomorphism $f\to f(z)$. It may be shown readily that the ideal (a) is an endo-ideal; in fact every ideal $(a-\lambda e)$ where $\lambda\in D$ is an endo-ideal. Hence by equation (5) we may write

$$f\sim\beta_0 e+\beta_1 a+\beta_2 a^2+\cdots. \tag{6}$$

The power series need not converge. (It is only in the ring $\mathfrak{R}\{a\}/\mathfrak{J}$ that a norm is defined for it.) A question which is raised immediately is: What is the relation between (6) and the Maclaurin expansion of $f(z)$? The answer is the expected one: The latter expansion is $f(z)=\beta_0+\beta_1 z+\beta_2 z^2+\cdots$. This fact is fundamental.

Now suppose that D is the largest domain in the spectrum of a which contains the origin. Consider for every λ in D the ideal $\mathfrak{J}_\lambda=\prod_{n=1}^{\infty}(a-\lambda e)^n$. Thus $\mathfrak{J}_\lambda$ consists precisely of all elements f such that in the expansion (5) in powers of $(a-\lambda e)$, each β_i is zero. Hence $f(z)=0$ in D. This means that $\mathfrak{J}_\lambda$ is independent of λ, $\mathfrak{J}_\lambda=\mathfrak{J}$. The principal theorem may now be stated.

THEOREM 6. *Let $\mathfrak{R}\{a\}$, D, and $\mathfrak{J}$ satisfy the conditions given above. Then the ring $\mathfrak{R}\{a\}/\mathfrak{J}$ is isomorphic to a subring of the ring of functions analytic in D and continuous over the closure of D. Under this isomorphism $a\to z$. If $\{f_n\}$ is a sequence such that $\lim_{n\to\infty} f_n=f$, then $\lim_{n\to\infty} f_n(z)=f(z)$ uniformly in D.*

This theorem states that $\mathfrak{R}\{a\}$ is locally (in the neighborhood of the origin) homomorphic to a ring of analytic functions. The norm dominating the ring of analytic functions is usually stronger than the ordinary uniform convergence norm. The ideal $\mathfrak{J}$ clearly contains the radical of $\mathfrak{R}\{a\}$. It also contains those elements generated by the connected components in the spectrum of a which are distinct from the component containing D. It is also clear that in $\bar{\mathfrak{R}}=\mathfrak{R}\{a\}/\mathfrak{J}$ the element $\bar{a}$ has for its spectrum at least the closure of D. Whether it can contain more is not known.

Rings of infinitely differentiable functions. The requirement that the spectrum of a contain a neighborhood will be lightened. In this section will be considered certain rings $\mathfrak{R}\{a\}$ where a has a real spectrum. More exactly the spectrum is to contain a certain real interval $[\alpha, \beta]$, $\beta-\alpha>0$, and furthermore, for every λ in this interval, the ideal $(a-\lambda e)$ is to be an endo-ideal. If $\{p_n(\lambda)\}$ is a sequence of polynomials such that $\lim_{n\to\infty} p_n(a)=f$ where f is some element in $\mathfrak{R}\{a\}$, then for any integer m

$$p_n(a) \equiv p_n(\lambda)e + p_n'(\lambda)(a-\lambda e) + \cdots + p_n^{(m)}(\lambda)(a-\lambda e)^m \mod (a-\lambda e)^{m+1}. \tag{7}$$

Conditions can now be enforced to guarantee that not merely $\lim_{n\to\infty} p_n(\lambda)=f(\lambda)$ but that for every $m\geqq 1$, $\lim_{n\to\infty} p_n^{(m)}(\lambda)=f^{(m)}(\lambda)$. It was seen earlier that for every λ and m there exists a projection $\mathfrak{R}\{a\}\to(a-\lambda e)^m$. The condition that will be imposed is that the bound of this projection be a constant M_m independent of λ on $[\alpha, \beta]$. Finally let $\mathfrak{Q}$ be the ideal $\mathfrak{Q}=\prod_{m,\lambda}(a-\lambda e)^m$ where the intersection is formed for all λ on $[\alpha, \beta]$ and all $m\geqq 1$. We are ready to state the principal result of this section.

THEOREM 7. *Let $\mathfrak{R}\{a\}$, $[\alpha, \beta]$, and $\mathfrak{Q}$ be as above. Let the projections $\mathfrak{R}\{a\}\to(a-\lambda e)^m$ be of bound M_m independent of λ. Then the ring $\overline{\mathfrak{R}}\{a\}=\mathfrak{R}\{a\}/\mathfrak{Q}$ is isomorphic to a subring of the ring of all functions infinitely differentiable on $[\alpha, \beta]$. The isomorphism carries a into x. If in $\mathfrak{R}\{a\}$, $\lim_{n\to\infty} f_n=f$ then uniformly on $[\alpha, \beta]$ and for $m\geqq 1$, $\lim_{n\to\infty} f_n^{(m)}(x)=f^{(m)}(x)$.*

The condition of quasi-analyticity of a ring may be formulated in terms of ideals in a simple fashion. A ring of functions is quasi-analytic on $[\alpha, \beta]$ if $f^{(m)}(\lambda)=0$, $m\geqq 0$, for some $\lambda\in[\alpha, \beta]$ implies $f(x)=0$ identically on $[\alpha, \beta]$. If we assume for the sake of simple notation that $\mathfrak{Q}$ in the above theorem is the zero ideal then we may say: $\mathfrak{R}\{a\}$ is quasi-analytic if and only if for some λ on $[\alpha, \beta]$, $\mathfrak{I}_\lambda=\prod_{n=1}^{\infty}(a-\lambda e)^n=0$. It may be quickly verified that if $\mathfrak{R}\{a\}$ is quasi-analytic, it is a normed power series ring. This may be seen from formula (5) and the discussion which follows it.

Absolute values of real elements. The formula (2) is the principal one for determining the reducibility of rings. In the exploitation of this formula, care must be taken that the curve C should not cross the spectrum of a. The question has been raised whether there are circumstances under which C is allowed to cross the spectrum. We shall answer this in the affirmative, not indeed for the integrand in

(2), but for a variation of it. Suppose for the sake of convenience that a has a real spectrum which includes an interval around $\lambda=0$. Suppose further that for all real $\alpha\neq 0$, for some $M>0$, and some real $s<1$, $|a(\alpha ie-a)^{-1}|\leqq|\alpha|^{-s}M$. Then the following improper integral exists:

$$b=\frac{1}{2\pi i}\int_C \frac{a}{\zeta e-a}\,d\zeta, \tag{8}$$

where C is a simple rectifiable curve cutting the axis of reals orthogonally, once at $\lambda=0$, and once more far to the right. For example C might be the circle $|\zeta-|a||=|a|$. It may be seen easily that the spectrum of b is real and non-negative and that $b(a-b)=0$. In fact we may say that $b=\max(a, 0)$ and we may define the absolute value of a by $|a|=2b-a$.

The condition $|a(\alpha ie-a)^{-1}|\leqq|\alpha|^{-s}M$ implies that (a) is an ecto-ideal. In fact the most frequently discussed cases of ecto-ideals seem to satisfy this inequality and hence lead to the notion of absolute value.

The use of the terminology $\max(a, 0)$ and $|a|$ may be justified as follows. According to Theorem 2, the element a in $\mathfrak{R}\{a\}$ corresponds to the function $f(\lambda)=\lambda$. If b corresponds to $g(\lambda)$, formula (8) modulo $(a-\lambda e)$ shows quickly that $b(\lambda)=\lambda$ if $\lambda\geqq 0$, $b(\lambda)=0$ if $\lambda<0$. Thus $b(\lambda)=\max(\lambda, 0)$. If we now assume that $\mathfrak{R}\{a\}$ has no radical, the correspondence of Theorem 2 is an isomorphism.

The principal ideals $(a-b)$ and (b) have no common elements, their product is zero and together they span the maximal ideal (a). In the cases usually considered they not only span (a), but their direct sum is (a).

These results may be used to prove the fundamental theorem concerning the structure of bounded self-adjoint transformations in Hilbert space. It is common knowledge that in establishing the equation

$$H=\int \lambda dE(\lambda), \tag{9}$$

no matter which of the many possible paths is taken, use has to be made at one or two critical points of powerful—if one prefers, subtle—tools whereas the remainder of the argument proceeds along more obvious lines. In the proof which we suggest these critical points are: (1) If H is self-adjoint and its spectrum is contained within the real interval $[\alpha, \beta]$, then $|H|=\max(|\alpha|, |\beta|)$ (or alternatively,

$\alpha \leqq (Hf, f) \leqq \beta$, $(f, f) = 1$). (2) If γ is an arbitrary point in $[\alpha, \beta]$ then there exist two self-adjoint transformations H_1 and H_2 in $\mathfrak{R}\{H\}$ such that the zero spaces of H_1 and H_2 form orthogonal manifolds whose direct sum is the Hilbert space $\mathfrak{H}$ and such that $H = H_1 + H_2$, $H_1 H_2 = 0$. The spectrum of H_1 lies essentially in $[\alpha, \gamma]$, that of H_2 in $[\gamma, \beta]$.

Proof of the first point is carried out with the help of an integral due to Hille which allows one to embed certain transformations with real non-negative spectrum in a semi-group.[26] The only transformation we need is $H^{1/2}$ in case $\alpha \geqq 0$. The second difficulty is surmounted by means of formula (8) in the manner which we have indicated.

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[26] See footnote 7, Theorem 14.

THE APRIL MEETING IN NEW YORK

The four hundred third meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, April 28–29, 1944. The attendance included the following one hundred seventy-nine members of the Society:

R. L. Anderson, R. G. Archibald, K. J. Arnold, L. A. Aroian, S. P. Avann, A. E. Basch, E. G. Begle, Stefan Bergman, Felix Bernstein, Garrett Birkhoff, G. D. Birkhoff, Gertrude Blanch, M. L. Boas, R. P. Boas, H. F. Bohnenblust, C. B. Boyer, H. W. Brinkmann, A. B. Brown, G. W. Brown, E. L. Buell, R. S. Burington, Hobart Bushey, J. H. Bushey, R. H. Cameron, W. B. Campbell, S. S. Chern, Edmund Churchill, R. V. Churchill, M. D. Clement, Nathaniel Coburn, Harvey Cohn, L. M. Comer, T. F. Cope, Richard Courant, A. P. Cowgill, W. H. H. Cowles, H. B. Curry, M. M. Day, Jesse Douglas, Nelson Dunford, W. L. Duren, William H. Durfee, J. E. Eaton, M. C. Eide, J. M. Feld, A. D. Fialkow, R. M. Foster, F. H. Fowler, G. A. Foyle, Bernard Friedman, K. O. Friedrichs, Orrin Frink, R. E. Fullerton, H. P. Geiringer, Abe Gelbart, David Gilbarg, B. P. Gill, Leonard Gillman, Michael Golomb, A. W. Goodman, R. O. Goodman, F. G. Gravalos, Bernard Greenspan, Lewis Greenwald, C. C. Grove, Laura Guggenbühl, E. J. Gumbel, Jacques Hadamard, P. R. Halmos, Olaf Helmer, A. D. Hestenes, M. R. Hestenes, L. S. Hill, Einar Hille, Abraham Hillman, Harold Hotelling, E. M. Hull, Ralph Hull, Witold Hurewicz, L. C. Hutchinson, B. M. Ingersoll, Florence Jacobson, Nathan Jacobson, S. A. Joffe, Fritz John, A. W. Jones, Aida Kalish, Edward Kasner, P. W. Ketchum, S. A. Kiss, J. R. Kline, E. G. Kogbetliantz, E. R. Kolchin, D. M. Krabill, M. E. Ladue, Howard Levene, Howard Levi, D. C. Lewis, Marie Litzinger, L. H. Loomis, E. R. Lorch, A. N. Lowan, N. H. McCoy, L. A. MacColl, J. K. L. MacDonald, Saunders MacLane, H. F. MacNeish, A. J. Maria, D. H. Maria, Carl Markow, W. T. Martin, D. S. Miller, L. W. Miller, E. C. Molina, Richard Morris, D. J. Morrow, F. J. Murray, C. A. Nelson, Otto Neugebauer, P. B. Norman, L. R. Norwood, C. O. Oakley, F. W. Owens, J. C. Oxtoby, E. L. Post, Walter Prenowitz, A. L. Putnam, Hans Rademacher, Mina Rees, Moses Richardson, R. G. D. Richardson, J. F. Ritt, R. K. Ritt, E. K. Ritter, P. C. Rosenbloom, Hans Samelson, Arthur Sard, L. J. Savage, R. H. Scanlan, Henry Scheffé, S. A. Schelkunoff, L. I. Schoenfeld, Alberta Schuettler, I. E. Segal, C. E. Shannon, Max Shiffman, J. A. Shohat, P. A. Smith, Fritz Steinhardt, J. J. Stoker, E. G. Straus, M. M. Sullivan, Fred Supnick, Otto Szász, J. D. Tamarkin, J. H. Taylor, J. M. Thomas, E. M. Torrance, H. F. Tuan, A. W. Tucker, J. W. Tukey, Oswald Veblen, R. M. Walter, W. R. Wasow, G. C. Webber, André Weil, Alexander Weinstein, Louis Weisner, C. P. Wells, George Whaples, A. P. Wheeler, W. L. G. Williams, John Williamson, H. P. Wirth, R. S. Wolfe, Jacob Wolfowitz, Oscar Zariski, M. F. Zucker, Antoni Zygmund.

On Friday evening at 8 o'clock Professor K. O. Friedrichs gave an address entitled *Mathematical aspects of the boundary layer theory*. Professor R. G. D. Richardson presided at this session.

On Saturday there were two addresses: at 11 A.M. *The structure of normed abelian rings* by Professor E. R. Lorch, and at 2 P.M. *Modern algebra and the Riemann hypothesis* by Professor André Weil. Pro-

fessor W. T. Martin presided at the first of these sessions, and Vice President Einar Hille at the second.

At 3:15 P.M. Saturday there was a session for papers in Algebra, Probability, Statistics and Applied Mathematics, at which Professor N. H. McCoy presided, and another for papers in Analysis and Geometry at which Professor Nelson Dunford presided.

The Council met at 12:30 P.M. on April 29, 1944, in the Long Room of the Faculty Club.

The Secretary announced the election of the following one hundred and twenty-seven persons to membership in the Society (the address given is that of the member at the time of his application):

Professor Louise Adams, High Point College, High Point, N. C.;
Dean Ralph Ellsworth Allison, Northwest Nazarene College, Nampa, Idaho;
Professor Hippocrates George Apostle, University of Rochester;
Dr. Edward George Baker, American Bureau of Shipping, New York, N. Y.;
Mr. Anthony Francis Bartholomay, Brown University;
Mr. Chester Edwin Baston, Hillyer Junior College, Hartford, Conn.;
Mr. Daniel Moore Bates, Philadelphia, Pa.;
Mr. William Edwin Beeman, North Texas State Teachers College, Denton Tex.;
Mr. John Bodnar, Junior College of Connecticut, Bridgeport, Conn.;
Mr. Claude Levi Bordner, State Teachers College, Kutztown, Pa.;
Professor Alger Vernon Boswell, Tennessee Agricultural and Industrial State Teachers College, Nashville, Tenn.;
Professor John Gerald Bowker, Middlebury College, Middlebury, Vt.;
Professor John Grist Brainerd, Moore School of Electrical Engineering, University of Pennsylvania;
Professor Angeline Jane Brandt, Wheaton College, Wheaton, Ill.;
Professor Robert H. Breusch, Amherst College;
Dr. Walter J. Bruns, Hamilton College;
Professor John Harvey Butchart, Grinnell College, Grinnell, Iowa;
Miss Minerva Elda Butz, Cedar Crest College, Allentown, Pa.;
Dr. Ralph Edward Byrne, Jr., Barber-Colman Company, Rockford, Ill.;
Mr. Joseph D. Chapline, Jr., Moore School of Electrical Engineering, University of Pennsylvania;
Mr. Eric H. Clamons, University of Minnesota;
Professor James P. Cole, Louisiana State University;
Captain Clarence Beasley Collier, Jr., United States Military Academy;
Sister Maria Loyola Conlan, College of Mount Saint Vincent, New York, N. Y.;
Mrs. F. Marion Clarke Connell (Mrs. James R.), University of Southern California;
Professor Erben Cook, Jr., University of Connecticut;
Mr. Irvin Douglas Cook, College Park, Md.;
Miss Nancy Elizabeth Cooley, Fairfax Hall Junior College, Waynesboro, Va.;
Mr. V. B. Cooper, South Georgia College, Douglas, Ga.;
Professor Forrest Eugene Craver, Dickinson College, Carlisle, Pa.;
Professor Don Cude, Southwest Texas State Teachers College, San Marcos, Tex.;
Mr. Leland E. Cunningham, Aberdeen, Md.;
Mr. John Moffatt Danskin, Jr., U. S. Naval Reserve, Wright Junior College, Chicago, Ill.;

Dr. Margaret Rae Davis, Greenbrier Junior College, Lewisburg, W. Va.;
Professor Lester C. Dawson, University of Alaska;
Dr. Frances Rousmaniere Dewing, Bennington College;
Sister Agnes Therese Dimond, College of Mount St. Joseph-on-the-Ohio, Mount St. Joseph, Ohio;
Mr. Harold Irving Ewen, Amherst College;
Mr. Edward Joseph Farrell, University of San Francisco;
Mr. David Fisher, Michigan College of Mining and Technology;
Professor Carroll E. Flanagan, State Teachers College, Superior, Wis.;
Mr. Frederick Fraser, St. Petersburg Junior College, St. Petersburg, Fla.;
Mr. Paul Thomas Gilbert, Jr., Utah State Agricultural College;
Mr. Alfred Oscar Ginkel, University of Rochester;
Mr. Eugene Van Ness Goetchius, University of Virginia;
Mr. Adolph W. Goodman, Syracuse University;
Dr. Fausto G. Gravalos, Goodyear Aircraft Company, Akron, Ohio;
Mr. Wallace Clayton Griffith, DePauw University;
Sister Edward Joseph Hackett, St. Mary's College, Holy Cross, Notre Dame, Ind.;
Mr. Franklin Haimo, Reed College;
Mr. Leon Albert Henkin, Kellex Corporation, New York, N. Y.;
Mr. Alvadore John Hinshaw, Central Union High School and Junior College, El Centro, Calif.;
Miss Lois Aileen Hostinsky, University of Illinois;
Reverend Joseph Albert Hratz, St. Ambrose College, Davenport, Iowa;
Mr. Stanley Parlett Hughart, Princeton University;
Dr. John Theodore Johnson, Chicago Teachers College;
Mr. Herbert L. Jones, Hiram College, Hiram, Ohio;
Mr. Winfield Keck, Muhlenberg College, Allentown, Pa.;
Mr. David B. Kirk, Haverford College;
Mr. Kenneth Virgil Knight, American Optical Company, Southbridge, Mass.;
Mr. John C. Koken, Parks Air College, East St. Louis, Ill.;
Mr. Yung-Huai Kuo, California Institute of Technology;
Dr. Paco Axel Lagerstrom, Princeton University;
Professor John Mark LaRue, Central Michigan College of Education, Mount Pleasant, Mich.;
Professor Herbert Leonard Lee, University of Tennessee;
Mr. Kenneth Levenberg, Frankford Arsenal, Philadelphia, Pa.;
Mr. William Judson LeVeque, University of Colorado;
Mr. Neil Lockwood, Michigan College of Mining and Technology;
Mr. Arthur John Lohwater, University of Rochester;
Miss Eleanor Ann Ludwig, Syracuse University;
Professor Clyde Truman McCormick, Fort Hays Kansas State College, Hays, Kan.;
Professor Edith Johnson McKissock, Youngstown College, Youngstown, Ohio;
Mr. Samuel Stuart McNeary, Drexel Institute of Technology;
Mr. Clarence Isaac McSwane, University of Illinois;
Dr. Murray Mannos, University of Delaware;
Mr. Carl Markow, St. Peter's College, Jersey City, N. J.;
Professor Milton Marshall, Brigham Young University;
Mr. Frank Harshman Miller, Kenyon College;
Mr. William H. Mills, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.;
Mr. Arthur Charles Moeller, Denison University, Granville, Ohio;

Mr. William Burton Moye, Georgia Teachers College;
Father Cronan Mullen, Siena College, Loudonville, N. Y.;
Mr. Allen Charles Munster, University of Pennsylvania;
Dr. Václav Myslivec, Czechoslovak Economic Service, New York, N. Y.;
Miss Elizabeth Ann Oliphant, Texas College of Arts and Industries, Kingsville, Tex.;
Professor James Enoch Parker, Knoxville College, Knoxville, Tenn.;
Professor Archie Edgar Patterson, School of Forestry, University of Georgia;
Professor Charles Roberts Pettis, Ohio State University;
Mr. William Augustus Pierce, University of Vermont;
Professor Raymond J. Pitts, Fort Valley State College, Fort Valley, Ga.;
Miss Helen Susanna Pollock, State University of Iowa;
Mr. David Pomeroy, Oregon State College;
Mr. Gordon Raisbeck, Stanford University;
Mr. Meyer Morris Resnikoff, Carleton College, Northfield, Minn.;
Mr. Charles Barton Reynolds, Federal Communications Commission, New York, N. Y.;
Mr. Robert Bruce Rice, Ohio State University;
Ensign Robert Errol Roberson, Naval Research Laboratory, Washington, D. C.;
Miss Helen Murray Roberts, University of Connecticut;
Mr. Gene Fuerst Rose, Glassport, Pa.;
Professor Francis Edward Rothchild, Arkansas Agricultural and Mechanical College, Monticello, Ark.;
Miss Margaret M. Ryan, Syracuse University;
Mr. Herbert Ellis Salzer, Mathematical Tables Project, National Bureau of Standards, New York, N. Y.;
Professor Charles Henry Sampson, Bates College;
Mr. George R. Schaefer, U. S. Naval Observatory, Washington, D. C.;
Sister M. Rosalin Schaeffer, Ursuline College, Louisville, Ky.;
Mr. Lowell I. Schoenfeld, Naval Air Experimental Station, Navy Yard, Philadelphia, Pa.;
Dr. Hans Wilhelm Eduard Schwerdtfeger, University of Adelaide, Adelaide, South Australia;
Lieutenant Commander Henry St. Clair Sharp, United States Coast Guard Academy;
Mr. William Lloyd Shepherd, Southwestern College, Winfield, Kan.;
Professor Heinz Simon, Southern Union College, Wadley, Ala.;
Mr. Frank Brooke Sloss, Westminster College, Fulton, Mo.;
Mr. Maurice Howard Slud, Brown University;
Professor Dudley Eugene South, University of Kentucky;
Miss Elisabeth Stanfield, Bennett College, Greensboro, N. C.;
Mr. Irwin Stoner, University of North Carolina;
Professor Earl Garrod Swafford, Fort Hays Kansas State College, Hays, Kan.;
Mr. John William Swank, Los Angeles, Calif.;
Dr. Robert Catesby Taliaferro, Hamilton College;
Mr. Francisco Vazquez Pueyo, Polytechnic Institute of Puerto Rico;
Mr. Anthony Emilio Ventriglia, Cornell University;
Professor George Frank Waites, Centenary College of Louisiana;
Professor Eleanor Boyd Walters, Delta State Teachers College, Cleveland, Miss.;
Mr. Dana Wallis Whitman, Burlington, Vt.;
Dr. Yung-Chow Wong, University of Pennsylvania;
Mr. Frederick Lewis Wood, State Teachers College, Bridgewater, Mass.;

Mrs. Mary Hart Zink (Mrs. F. J.), Elmhurst College, Elmhurst, Ill.;
Mrs. Miriam F. Zucker (Mrs. S. M.), American Mathematical Society, New York, N. Y.

It was reported that the following had been elected as nominees on the Institutional Memberships of the Institutions indicated:

Brown University: Messrs. Joaquín Basilio Díaz and George Herman Handelman.
Bryn Mawr College: Miss Marie Anna Wurster.
University of California: Miss Louise Hoy Chin, Messrs. E. Allen Davis, Joseph Lawson Hodges, and Siegfried Friedrich Neustadter.
University of California at Los Angeles: Mr. Gordon Klinzman Overholtzer, Mrs. Zivia Wurtele.
University of Chicago: Messrs. Charles F. Brumfiel, Marvin Maynard Lavin, and Charles A. Nichols.
University of Cincinnati: Mr. John Sunapee Stubbe.
Cornell University: Messrs. Robert Lewis Beinert, Charles Hatfield, Jr., and Edward Joseph Scott.
Duke University: Mr. Robert Arthur Clark.
Harvard University: Messrs. E. J. Akutowicz, Philip Davis, L. Fine, Daniel Gorenstein, A. R. Harvey, I. I. Hirschman, Robert Heywood Hoskins, G. Mostow, S. B. Sommerville, James Wolfe, and Joseph Abraham Zilber.
University of Illinois: Miss Mildred Jeannette Brannon, Mr. H. Herbert Fox, Misses Helen Louise Friend, Helen P. Hoffman, and Elaine Vivian Nantkes, Mr. Richard George Stoneham, Professor Earl Reeves Wasserman.
Iowa State College: Mr. Frank Edward Bortle.
Johns Hopkins University: Messrs. Thomas Birdsal Andrews, Jr., Charles Vale Bitterli and Julius Vogel.
Massachusetts Institute of Technology: Messrs. Lawrence Russell Norwood, Walter Pitts, and George Neal Raney.
University of Michigan: Messrs. Pratap Chand and Manuel O. Hizon, Miss Frances Van Every Morfoot.
Northwestern University: Mr. William R. Allen, Miss Alice Christianson, Messrs. B. Kent Dickerson and Martinus Esser, Miss Evelyn Frank.
University of Notre Dame: Rev. Ferdinand L. Brown.
Pennsylvania State College: Mr. John Rankin Kinney.
Stanford University: Miss Kathryn Anne Morgan.
Syracuse University: Professor Howard Whitley Eves.
Texas Technological College: Miss Virginia Elizabeth Bowman.
University of Virginia: Messrs. Raymond Alfred Lytle and Russell Lowell Wine.
University of Washington: Miss Phyllis Cady Johnson, Mr. Max Carl Walske.
Wesleyan University: Messrs. Adolph Grünbaum and Raymond de Shelly.
Western and Southern Life Insurance Company: Mr. William S. Marcaccio.
Williams College: Mr. Edwin Foote Gillette.
University of Wisconsin: Messrs. Louis Harold Kanter, John Thomas Moore, Joseph Rosenthal, Raphael Darrel Wagner, and Robert Livingston Young.
Yale University: Mr. Walter John Klimczak.

It was also reported that Immaculata College, Immaculata, Pennsylvania, had been elected to institutional contributing membership in the Society.

The following appointments by President M. H. Stone were reported: as a committee to study the present method of arranging scientific programs, Professors Saunders MacLane (Chairman), W. T. Martin, C. V. Newsom, Gabor Szegö, and Oswald Veblen; as a Committee on Nomination of Officers and Members of the Council for 1945, Professors Einar Hille (Chairman), G. A. Hedlund, E. P. Lane, C. C. MacDuffee, and Hassler Whitney; as a Committee on Arrangements for the 1944 Summer Meeting at Wellesley College, Professor Marion E. Stark (Chairman), Dr. Katherine E. Hazard, Professors T. R. Hollcroft, B. W. Jones, and Helen G. Russell; as auditors of the Society's accounts for 1944, Professors T. F. Cope and A. E. Meder, Jr.; as representative of the Society at the inauguration of Clark George Kuebler as seventh President of Ripon College (Ripon, Wisconsin) on December 7, 1943, Professor R. E. Langer; as representative of the Society at the inauguration of Russell David Cole as President of Cornell College (Mount Vernon, Iowa) on April 25, 1944, Professor Roscoe Woods; as representative of the Society at the inauguration of the Very Reverend Vincent J. Flynn as President of The College of St. Thomas (St. Paul, Minnesota) on April 27, 1944, Professor R. W. Brink.

The Secretary reported that the following had been chosen as the voting representatives of editorial committees on the Council for the year 1944: Bulletin—Dean Tomlinson Fort; Transactions—Professor A. A. Albert; Colloquium Publications—Professor A. B. Coble; American Journal of Mathematics—Professor G. D. Birkhoff; Mathematical Reviews—Professor O. E. Neugebauer.

The Secretary reported that Professor John von Neumann had accepted the invitation to deliver the Gibbs Lecture at the Annual meeting of 1944; the title of the Lecture will be *The ergodic theorem and statistical mechanics.*

It was reported that the biennial List of Members, usually published as a supplement to the September Bulletin, will be published in November in 1944.

The Secretary reported the receipt of a letter from the London Mathematical Society, expressing appreciation of the fifteen microfilm reading machines, donated by our Society, which have now been distributed to scientific groups in Great Britain.

The Council adopted the following resolution on the death of Professor Thomas Scott Fiske:

The death on January 10, 1944, of Professor Thomas Scott Fiske, venerable founder of the American Mathematical Society, brought deep sadness to the country's mathematicians.

To the members of the Society, the story of its first years is an absorbing one. In 1888, Professor Fiske, then a young man of twenty-three, saw that the time was ripe for the organization of the mathematicians of America. Through his efforts there was established the New York Mathematical Society which, in the space of three years, grew into a national organization. Fiske held office in the Society as Secretary from 1888 to 1895, as Treasurer from 1898 to 1901 and, finally, as President, 1903 to 1904. In addition, he served for long periods on the editorial staffs of the Bulletin and Transactions.

Fiske's ability as an organizer also found scope in the Department of Mathematics of Columbia University, of which he was a member for fifty years. Himself a scholar of genuine depth, he devoted his energies to the creation of a center of mathematical research at Columbia. He took a sincere interest in the researches of his colleagues and found particular pleasure in encouraging the scientific efforts of the young men serving under him.

Fiske was a striking personality. Distinguished in presence, cordial and frank in nature, he enjoyed the deep regard of his associates. He has left his mark on a period of mathematical history whose color will deepen as the years advance.

On recommendation of the Committee on Places of Meetings, the 1944 Annual Meeting was scheduled for Chicago, Illinois, on November 24–25. The Council also scheduled a far western meeting on November 25 at the University of California at Los Angeles.

It was reported that Professor Will Feller had been invited to deliver an hour address at the 1944 Annual Meeting.

Professors F. J. Murray and Morgan Ward were reappointed as representatives of the Society on the Editorial Board of the Duke Mathematical Journal.

The Council accepted the resignation of Professor J. W. T. Youngs as Acting Associate Secretary.

The Council voted to recommend to the Board of Trustees that the Transactions of the Society be copyrighted.

The Secretary reported for the War Policy Committee as follows: (1) Problems in connection with mathematicians and Selective Service have occupied the attention of members of the Committee and the Secretary's office has attempted to keep chairmen of departments of mathematics informed of changes in procedure; (2) Efforts are being made to secure funds so that the Society and Association may continue to support the Office of Scientific Personnel in Washington, the Director of which has been useful in representing the mathematicians during these critical times; (3) the Committee on Available Teachers of Collegiate Mathematics is continuing to function as an aid to department chairmen in securing teachers for the service programs; (4) the Subcommittee on War Training Programs has submitted detailed reports on the mathematical aspects

of the Army Specialized Training Program and of the Navy College Training Program. The Subcommittee is also studying the work of the Armed Forces Institute.

On recommendation of the committee appointed to consider the possibility of changing the term of office of the President of the Society, the Council voted that no change be made at this time.

Titles and cross references to the abstracts of papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers numbered 1 to 6 were read in the section for Algebra, Probability, Statistics, and Applied Mathematics; papers 7 to 12 in the section for Algebra and Geometry; papers 13 to 28 were read by title. Paper 1 was read by Professor Tukey; paper 9 by Professor Cameron. Dr. Kober was introduced by Professor Einar Hille.

1. Henry Scheffé and J. W. Tukey: *Contributions to the theory of non-parametric estimation.* (Abstract 50-5-164.)

2. John Williamson: *Hadamard's determinant theorem and the sum of four squares.* (Abstract 50-5-119.)

3. Garrett Birkhoff: *Subdirect products in universal algebra.* (Abstract 50-5-106.)

4. J. A. Shohat: *Series expansions for the periodic solution of Van der Pol's equation and its frequency for all values of the parameter.* (Abstract 50-5-156.)

5. P. R. Halmos: *Random alms.* (Abstract 50-5-163.)

6. S. P. Avann: *Relations between join-irreducibles and meet-irreducibles in a modular lattice.* I. (Abstract 50-5-104.)

7. L. H. Loomis: *Abstract congruence and the uniqueness of Haar measure.* (Abstract 50-5-141.)

8. S. S. Chern: *Laplace transforms of a class of higher dimensional varieties in a projective space of n dimensions.* (Abstract 50-5-159.)

9. R. H. Cameron and W. T. Martin: *Transformations of Wiener integrals under a general class of linear transformations.* (Abstract 50-5-130.)

10. Otto Szász: *On uniform convergence of trigonometric series.* (Abstract 50-5-149.)

11. L. I. Schoenfeld: *A transformation formula in the theory of partitions.* (Abstract 50-5-147.)

12. M. M. Day: *On symmetric closed convex curves.* (Abstract 50-5-132.)

13. R. P. Agnew: *Summability of subsequences.* (Abstract 50-5-120-*t*.)

14. Stefan Bergman: *Certain classes of analytic functions of two real variables and their properties.* (Abstract 50-5-123-*t*.)

15. Stefan Bergman: *The determination of some properties of a function satisfying a partial differential equation from its series development.* (Abstract 50-5-124-*t*.)

16. Augusto Bobonis: *A sufficiency theorem for differential systems.* (Abstract 50-5-126-*t*.)

17. H. D. Brunk: *Theorems of composition for Dirichlet series.* Preliminary report. (Abstract 50-5-128-*t*.)

18. C. W. Churchman and Benjamin Epstein: *Estimates of error in parallel experiments.* (Abstract 50-5-162-*t*.)

19. W. B. Fite: *The degree of a linear homogeneous group.* (Abstract 50-5-109-*t*.)

20. Michael Goldberg: *A three-space analog of a plane Kempe linkage.*) (Abstract 50-3-81-*t*.)

21. S. B. Jackson: *Vertices of plane curves.* (Abstract 50-5-160-*t*.)

22. Edward Kasner and John DeCicco: *Isothermal families in general curvilinear coordinates, and loxodromes.* (Abstract 50-7-191-*t*.)

23. Herman Kober: *Approximation by integral functions in the complex domain.* (Abstract 50-5-140-*t*.)

24. A. N. Lowan and H. E. Salzer: *Coefficients for complex interpolation within a square grid.* (Abstract 50-5-142-*t*.)

25. G. W. Mackey: *On infinite dimensional linear spaces.* (Abstract 50-5-143-*t*.)

26. A. M. Peiser: *On the average sum of the real roots of a random algebraic equation.* (Abstract 50-5-144-*t*.)

27. G. T. Whyburn: *Interior mappings into the circle.* (Abstract 50-7-197-*t*.)

28. Oscar Zariski: *The theorem of Bertini concerning the variable singular points of a linear system of varieties.* (Abstract 50-7-194-*t*.)

J. R. KLINE,
Secretary

THE APRIL MEETING IN CHICAGO

The four hundred fourth meeting of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, on Friday and Saturday, April 28–29, 1944. The attendance was about one hundred twenty-five, including the following one hundred eleven members of the Society:

A. A. Albert, Warren Ambrose, E. W. Anderson, Reinhold Baer, Walter Bartky, P. O. Bell, S. F. Bibb, K. E. Bisshopp, B. H. Bissinger, G. A. Bliss, D. G. Bourgin, A. J. Brandt, J. L. Brenner, R. H. Bruck, Herbert Busemann, L. E. Bush, F. A. Butter, R. E. Byrne, Abraham Charnes, L. W. Cohen, Nancy Cole, Max Coral, B. K. Dickerson, Samuel Eilenberg, H. P. Evans, C. J. Everett, H. S. Everett, L. R. Ford, J. S. Frame, Evelyn Frank, J. H. Giese, J. W. Givens, R. A. Good, Cornelius Gouwens, L. M. Graves, L. W. Griffiths, V. G. Grove, R. W. Hamming, W. L. Hart, M. L. Hartung, C. T. Hazard, O. C. Hazlett, M. H. Heins, E. D. Hellinger, Fritz Herzog, J. F. Heyda, E. H. C. Hildebrandt, T. H. Hildebrandt, J. D. Hill, D. L. Holl, A. S. Householder, H. K. Hughes, M. H. Ingraham, Karl Johannes, G. K. Kalisch, Wilfred Kaplan, William Karush, D. E. Kearney, Fred Kiokemeister, W. C. Krathwohl, R. E. Langer, E. H. Larguier, A. L. Lewis, M. I. Logsdon, A. T. Lonseth, W. C. McDaniel, Janet MacDonald, C. C. MacDuffee, Morris Marden, J. R. Mayor, C. W. Mendel, Karl Menger, H. J. Miser, C. W. Moran, E. J. Moulton, A. L. Nelson, C. J. Nesbitt, Ivan Niven, Isaac Opatowski, J. W. Porter, Tibor Rado, R. B. Rasmusen, Maxwell Reade, W. T. Reid, Haim Reingold, M. D. P. Rochford, R. G. Sanger, O. F. G. Schilling, G. E. Schweigert, M. E. Shanks, I. S. Sokolnikoff, Abraham Spitzbart, N. E. Steenrod, C. F. Stephens, B. M. Stewart, A. H. Stone, R. L. Swain, J. L. Synge, R. M. Thrall, M. K. Toft, Henry Van Engen, Bernard Vinograde, S. E. Warschawski, J. V. Wehausen, G. W. Whitehead, L. R. Wilcox, J. E. Wilkins, M. A. Wurster, G. S. Young, J. W. T. Youngs, M. H. Zink.

Sessions for the reading of contributed papers were held on Friday morning, with Professor J. S. Frame presiding, and Saturday morning with Professor J. L. Synge presiding. On Friday afternoon, Professor J. L. Synge gave an invited address on *The problem of Saint Venant for a cylinder with free sides*, and Professor I. S. Sokolnikoff gave an invited address on *Torsion and flexure of compound prisms*. Professor Walter Bartky presided at these lectures.

All sessions were held in the West lecture hall in the Museum of Science and Industry.

Titles and cross references for the abstracts of papers read at the meeting follow below: Papers 1 to 8 were read Friday morning, papers 9 to 17 on Saturday morning, papers 18 to 38, whose abstract numbers are followed by the letter *t* were read by title. Paper 2 was read by Dr. Bissinger and paper 5 by Dr. Kaplan. Ida Roettinger was introduced by Professor R. V. Churchill.

1. J. E. Wilkins: *The contact of a cubic surface with a ruled surface.* (Abstract 50-5-161.)

2. B. H. Bissinger and Fritz Herzog: *An extension of some previous results on generalized continued fractions.* (Abstract 50-5-125.)

3. S. E. Warschawski: *On conformal mapping of nearly circular regions.* (Abstract 50-5-151.)

4. Maxwell Reade: *On functions of class PL.* (Abstract 50-5-145.)

5. Wilfred Kaplan and Max Dresden: *The mechanism of the condensation of gases.* (Abstract 50-5-152.)

6. Morris Marden: *Axisymmetric harmonic vectors.* (Abstract 50-5-153.)

7. H. K. Hughes: *The asymptotic expansions of entire functions defined by Maclaurin series.* (Abstract 50-5-138.)

8. Tibor Rado: *On the Geöcze area of Frechet surfaces.* Preliminary report. (Abstract 50-7-182.)

9. R. H. Bruck: *Quasigroups with the inverse property.* I. Preliminary report. (Abstract 50-5-107.)

10. G. K. Kalisch: *Generalized Hilbert spaces over fields with a non-Archimedean valuation.* (Abstract 50-5-113.)

11. G. S. Young: *Concerning spaces in which every arc has two sides.* (Abstract 50-5-170.)

12. Samuel Eilenberg: *Skew-invariant cohomology groups.* Prelininary report. (Abstract 50-5-166.)

13. N. E. Steenrod: *The classification of sphere bundles.* (Abstract 50-5-168.)

14. P. O. Bell: *A study of surfaces by means of a system of differential equations of the first order.* (Abstract 50-5-158.)

15. R. M. Thrall: *On modular representations induced by ordinary representations.* (Abstract 50-5-117.)

16. Bernard Vinograde: *On properties of completely primary rings.* (Abstract 50-5-118.)

17. Warren Ambrose: *A variation of a theorem of I.E. Segal on the group ring of a compact group.* (Abstract 50-7-172-*t*.)

18. Leon Alaoglu and Paul Erdös: *On highly composite and similar numbers.* (Abstract 50-5-103-*t*.)

19. Reinhold Baer: *Groups without proper isomorphic quotient groups.* (Abstract 50-5-105-*t*.)

20. Reinhold Baer: *Crossed isomorphisms.* (Abstract 50-7-173-*t*.)

21. Reinhold Baer: *The fundamental theorems of elementary geometry. An axiomatic analysis.* (Abstract 50-5-157-*t*.)

22. E. F. Beckenbach and Maxwell Reade: *Further results on mean-values and harmonic polynomials.* (Abstract 50-5-121-*t*.)

23. E. F. Beckenbach and Maxwell Reade: *Regular solids and harmonic polynomials.* (Abstract 50-5-122-*t*.)

24. R. H. Bruck: *Simple quasigroups.* (Abstract 50-5-108-*t*.)

25. H. D. Brunk: *Transformations of Dirichlet series.* Preliminary report. (Abstract 50-5-129-*t*.)

26. M. M. Day: *Cluster points of subsequences.* (Abstract 50-5-131-*t*.)

27. M. M. Day: *Inner-product spaces.* (Abstract 50-5-133-*t*.)

28. J. D. Hill: *Cesàro summability of sequences of* 0*'s and* 1*'s.* (Abstract 50-5-136-*t*.)

29. J. D. Hill: *Nörlund methods of summability that include the Cesàro methods of all positive orders.* (Abstract 50-5-137-*t*.)

30. G. B. Huff: *The completion of a theorem of Kantor.* (Abstract 50-3-93-*t*.)

31. Albert Newhouse: *On finite extending groups.* (Abstract 50-5-115-*t*.)

32. Maxwell Reade: *Two applications of generalized Laplaceans.* (Abstract 50-5-146-*t*.)

33. Ida Roettinger: *Certain integral transformations, with applications to boundary value problems.* (Abstract 50-5-155-*t*.)

34. M. F. Smiley: *An application of lattice theory to quasigroups.* (Abstract 50-5-116-*t*.)

35. C. F. Stephens: *Nonlinear difference equations analytic in a parameter.* (Abstract 50-5-148-*t*.)

36. F. A. Valentine: *Contractions in non-Euclidean spaces.* (Abstract 50-5-150-*t*.)

37. G. S. Young: *A generalization of Moore's theorem on simple triods.* (Abstract 50-5-169-*t*.)

38. G. S. Young: *On continua whose links are non-intersecting.* (Abstract 50-5-171-*t*.)

J. W. T. YOUNGS,
Acting Associate Secretary

THE APRIL MEETING IN BERKELEY

The four hundred fifth meeting of the American Mathematical Society was held at the University of California on Saturday, April 29, 1944. The attendance was about fifty-five, including the following thirty-eight members of the Society:

H. L. Alder, H. M. Bacon, E. W. Barankin, B. A. Bernstein, W. Z. Chien, L. H. Chin, C. H. Dix, Herbert Federer, A. L. Foster, S. M. Hallam, O. H. Hamilton, J. G. Herriot, Alfred Horn, D. H. Hyers, E. L. Lehmann, D. H. Lehmer, S. H. Levy, J. V. Lewis, C. C. Lin, A. D. Michal, A. P. Morse, S. F. Neustadter, A. R. Noble, George Polya, J. B. Robinson, R. M. Robinson, R. A. Rosenbaum, A. C. Schaeffer, S. A. Schaaf, D. C. Spencer, Pauline Sperry, L. H. Swinford, Gabor Szegö, Alfred Tarski, R. K. Wakerling, L. F. Walton, A. R. Williams, František Wolf.

After a brief opening session for contributed papers at which Professor B. A. Bernstein presided, the meeting in the morning was devoted to the invited one hour addresses at which Professor Gabor Szegö presided. By invitation of the Program Committee, Professor D. H. Hyers of the University of Southern California and the California Institute of Technology spoke on *Linear topological spaces* and Professor R. M. Robinson of the University of California on *Hadamard's three circles theorem.*

The Symposium on Applied Mathematics was held in the afternoon with Professor George Polya presiding. The program consisted of two addresses: *Hydrodynamical stability*, by Dr. C. C. Lin, and *The intrinsic theory of elastic plates and shells*, by Dr. W. Z. Chien, both of the California Institute of Technology.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers 1 to 3 were read in the morning session while papers 4 to 11 were read by title. Paper 3 was read by Professor Foster. Mr. Reed was introduced by Professor Michal.

1. J. G. Herriot: *Nörlund summability of multiple Fourier series.* (Abstract 50-5-135.)

2. Herbert Federer: *The Gauss-Green theorem.* (Abstract 50-5-134.)

3. A. L. Foster and B. A. Bernstein: *A dual-symmetric definition of field.* (Abstract 50-5-110.)

4. E. F. Beckenbach and R. H. Bing: *On generalized convex functions.* (Abstract 50-5-165-*t*.)

5. H. D. Brunk: *Dirichlet series meromorphic in a half-plane.* Preliminary report. (Abstract 50-5-127-*t*.)

6. R. C. James: *Linear functionals and orthogonality in normed linear spaces.* (Abstract 50-5-167-*t*.)

7. R. D. James: *Power series expansions for doubly periodic functions of the second kind.* Preliminary report. (Abstract 50-5-111-*t*.)

8. S. A. Jennings: *A note on chain conditions in nilpotent rings and groups.* (Abstract 50-5-112-*t*.)

9. A. D. Michal: *Physical models of some curved differential-geometric metric spaces of infinite dimensions.* II. *Vibrations of elastic beams and other elastic media as studies in geodesics.* Preliminary report. (Abstract 50-5-154-*t*.)

10. Knox Millsaps: *On powers of third and fourth order matrices.* (Abstract 50-5-114-*t*.)

11. I. S. Reed: *On the solution of a general transform.* (Abstract 50-7-181-*t*.)

A. D. MICHAL,
Associate Secretary

OSKAR BOLZA—IN MEMORIAM

Oskar Bolza came to the United States from Germany in 1888, and returned in 1910 to make his home in Freiburg im Baden during the later years of his life. He was an influential member of our American mathematical community for twenty-two years, including those significant ones during which the American Mathematical Society was founded and had its early growth. He was one of the founders of the Chicago Section of the Society, was a member of the Council of the Society in 1900–1902 and vice president in 1904. He was beloved by his students and associates. He was born on May 12, 1857 in Bergzabern, Germany, and word reached this country in the autumn of 1942 through the American Red Cross that he died peacefully on July 5, 1942, presumably in Freiburg. The following pages are devoted to a sketch of his life and scientific activities. They have unusual significance for our American mathematical community.

Bolza's father was in the judicial service in Germany and the family dwelt in various places in southern Germany during Bolza's childhood years. In 1873 they made Freiburg their permanent home, and the ties which bound Bolza to that city were thereafter always very strong. His mother was a daughter of Friedrich Koenig, the inventor of a rapid printing press in the early part of the nineteenth century. The manufacture of printing presses proved to be profitable through the years, and so far as is known Bolza's life was free from financial worries.

Even in his early years Bolza was an industrious and independent student. His first strong interests seemed to be the beginning of a career in languages and comparative philology. But in 1873 he attended an academy in Neuchatel and later the Gymnasium in Freiburg, in both of which he had excellent instructors in mathematics. There for the first time he became enamoured of the subject, and mathematics remained his principal interest during all but a few years of the rest of his life.

In the autumn of 1875 Bolza went to Berlin to begin his university student career. At that time it was expected that he would ultimately succeed to a share in the management of the family printing press factory. But his father wanted him to have a liberal as well as a technical education, and for that reason he first attended both the University and the Gewerbeakademie in Berlin. It soon became clear that this double program was impractical, and that Bolza's interests were likely to lie much more in pure science than in business. He at first

turned to physics as a possible principal life activity, and during the years 1876–1878 he studied in that field under Kirchoff and Helmholz in Berlin, Quinke in Heidelberg, and Kundt in Strassburg. He even undertook to write a doctor's dissertation in physics under Kundt. But experimental work had always been burdensome to him and in the autumn of 1878 he decided finally, and certainly fortunately in view of his later attainments, to make pure mathematics his specialty.

The next three years Bolza pursued his interest in pure mathematics under Christoffel and Reye in Strassburg, Weierstrass in Berlin, Schwarz in Göttingen, and Weierstrass again. He was not pressed financially and had the approval of his father in his effort to prepare himself for a university career. But in the autumn of 1880 he found himself still far from his doctor's degree and already in his eleventh university semester. He determined therefore to prepare himself for the Staatsexamen which in Germany was a prerequisite for Gymnasium teaching though not for a university career. It was three years before his examination was passed and his year of practice teaching in the Gymnasium at Freiburg completed. He did well in this undertaking, and was successful and greatly interested in his teaching work.

For two years, 1883–1885, he worked privately on his doctor's dissertation in Freiburg. The subject which he had chosen was the determination of hyperelliptic integrals which can be reduced to elliptic integrals by a transformation of the third degree. He found the solution of his problem in 1885 just as a more elegant one by Goursat appeared in the Comptes Rendus. Nothing daunted Bolza attempted and found it easy to solve the corresponding problem for transformations of the fourth degree. His solution was published promptly in the Berichten der Freiburger Naturforschenden Gesellschaft. It was his first published paper.

In 1885 and 1886 Bolza studied again at Berlin under Fuchs and Kronecker. He extended his results on the transformation problem and after some correspondence with Schwarz and Klein in Göttingen his paper was accepted by the latter as a dissertation for the doctor's degree. Bolza's results were undoubtedly of great interest to Klein since they were closely related to some which Klein had himself recently published. In June 1886 Bolza passed his examination and was awarded the doctor's degree at the age of 29.

The next year, 1886–1887, Bolza and his intimate student friend Heinrich Maschke had a kind of private seminar with Klein in Göttingen. The three met once a week at Klein's house and planned and reported on research projects of common interest. It was a great experience for the younger men, but it had an unexpected effect upon

Bolza. Klein had an altogether exceptional genius and breadth of view in mathematics, and his personality was justifiably confident and overpowering. Bolza was a modest person, and as a result of his association with Klein he experienced a loss of confidence in himself which was completely unjustified, as later years have well shown.

It was at this stage in his career that Bolza debated seriously with himself what his future might turn out to be. As a student in Berlin he had made two exceptionally intimate young friends, the mathematician Heinrich Maschke mentioned above and the physicist Franz Schulze-Berge. Both of them had taken positions as gymnasium instructors but had found the teaching exceedingly heavy with little opportunity for scientific work. In the spring of 1887 Schulze-Berge decided to try his fortune in the United States, and upon his arrival here he promptly found a place in the experimental laboratory of Thomas Edison in New Jersey. This experience and the persuasiveness of two American mathematical students in Göttingen, M. W. Haskell and F. N. Cole, decided Bolza to follow the example of his friend, and in the spring of 1888 he landed in Hoboken.

Not long after his arrival he was greatly encouraged to find a place as "Reader in Mathematics" at Johns Hopkins University beginning in January 1889. Before the college year was out he was appointed "Associate in Mathematics" in the newly founded Clark University of Worcester, Massachusetts. The appointment began in October 1889 when the University first opened. Clark University had been founded as a graduate school with funds which soon proved to be insufficient. The faculty became restive and dissatisfied, and in 1892 when the new University of Chicago was opened, President Harper was able to persuade a number of them to move to his still newer institution. Bolza was one of these. He was made an associate professor, beginning January 1, 1893, with the understanding that he should be advanced to a full professorship after one year. He had also persuaded President Harper and E. H. Moore, the youthful head of the department of mathematics, to appoint Maschke to an associate professorship, Maschke having meanwhile also come to the United States.

Bolza was highly successful in his teaching and research during the eighteen years which he spent at the University of Chicago. The trio, Moore, Bolza, Maschke, had a profoundly stimulating influence on each other and on the continually numerous group of graduate students who went to Chicago to listen to them, a group now widely distributed among the faculties of our American universities. Among notable mathematical events which took place during this period were

the International Mathematical Congress at the World's Fair in Chicago in 1893 and the immediately following inspiring Colloquium Lectures by Klein at Northwestern University, the transformation of the New York Mathematical Society into the American Mathematical Society, the foundation of the Chicago Section and other sections of the Society, and the foundation of the Transactions of the American Mathematical Society in 1900. In all of these activities Moore, Bolza, and Maschke took a prominent part. At the third colloquium of the Society in Ithaca in August 1901, Bolza lectured in beautiful fashion on the calculus of variations, a subject then much in vogue because of the then recent appearance of Kneser's book in that field.

In March of 1908 Maschke died and one of the strongest ties which bound Bolza to Chicago was thereby broken. During his sojourn in Chicago Bolza had frequently refreshed his relations with his relatives and friends in Germany by journeys to Freiburg during his vacation periods. Furthermore he had the feeling that he should make way for some of the large group of young and able American mathematicians then becoming established in increasing numbers in American mathematical departments. Soon after Maschke's death, therefore, Bolza decided to return to Freiburg to spend his later years in his original home environment. The mathematical department at the University of Chicago released him with great reluctance and with the title Nonresident Professor of Mathematics which he held until his death.

In Freiburg Bolza was appointed Ordentlicher Honorar professor at the University there, at the instigation of his friend, Lüroth, and for several years he continued his mathematical lectures and research with great success. In the summer quarter of 1913 he lectured again at the University of Chicago to the great satisfaction of his friends and hearers. But the war of 1914–1918 had a disastrous effect upon Bolza's mathematical research. He continued his lectures until 1926 but from as early as 1917, under the influence of several very good friends, his interest turned strongly to religious psychology and to Sanskrit which he studied for the purpose of reading at first hand literature concerning the religious systems of India. The result of these activities was a book entitled *Glaubenlose Religion* published in 1930 under the pseudonym F. H. Marneck.

After an interruption of three years Bolza resumed his mathematical lectures in 1929 at the University of Freiburg before the largest classes he had ever had. In one case ninety students listened to his lectures on differential equations. But the flood of students ebbed away there as elsewhere during succeeding years, and in 1933 at the

age of 76 Bolza finally gave up lecturing and turned again to his psychological studies. Then a very interesting thing happened to him. Professor J. H. MacDonald, one of his Ph.D. students in 1901, sojourned for two weeks in Freiburg, and the two of them took long walks together during which they discussed again the subject which had long before engaged their interest, the transformation of hyperelliptic into elliptic integrals by transformations of specified degrees. Bolza was inspired to attempt anew some research in this field of his earliest mathematical activity and he succeeded in completing the solution of a problem which had long interested him much. His results were published in the Mathematische Annalen in 1935 in the paper numbered 57 in the bibliography below. In 1938 Bolza published one further study in religious psychology (no. 62 in the bibliography). It was his last published paper.

We know, however, that Bolza's enthusiasm for mathematics continued to the end. In a letter to Professor Arnold Dresden, written in 1939 at the age of 82, Bolza tells of his interest in studying the foundations of geometry with his friend and colleague in Freiburg, the geometer Heffter. According to Bolza's own statement his intention was to try to fill a yawning gap in his earlier mathematical training.

Since 1939 we have had little news from Bolza until the announcement in the autumn of 1942 of his death. But the impressions of his able and kindly personality which he has left behind, and the records of his inspiring lectures and research, are a part of our American mathematical traditions which will be with us always.

In the pages to follow in this biography some indications will be given of the scope and character of Bolza's scientific work.

The Scientific Work of Oskar Bolza

The scope and emphasis of Bolza's research are indicated roughly by the following table which lists for each of his principal fields the papers which he published and the years during which he was active in that field. The numbers refer to the bibliography at the end of this paper.

I. Elliptic and hyperelliptic integrals and functions: 1–6, 13–15, 17–20, 22, 23, 25, 60; 1885–1901, 1936.
II. Miscellaneous: 7–12, 16, 21, 34, 40, 41, 58, 61, 62; 1888–1938.
III. Calculus of variations: 24, 26–33, 35–39, 42–53, 56, 57, 59; 1901–1933.
IV. Integral equations and general analysis: 46, 54, 55; 1910–1914.

In Group I Bolza studied first the problem of determining hyper-

elliptic integrals which are reducible to elliptic integrals by transformations especially of the third and fourth degrees. It was his first research interest, the field of his dissertation sponsored by Klein in Göttingen, a problem often studied by leading mathematicians in the latter part of last century. Beginning with his sojourn under Klein in Göttingen in 1886, and continuing during something more than a decade following his move to the United States, Bolza published eleven papers concerning hyperelliptic Θ-, σ-, and $\wp$-functions and related subjects. In these he emphasized the elliptic theory and reformulated it in numerous instances to show how it appears as a special case of the hyperelliptic theory, and he carried over to the more general case many theorems and formulas well established in connection with the theory of elliptic functions. He was an able follower of his great teachers, Weierstrass and Klein, in this field.

When Bolza first went to Johns Hopkins in 1889 he was asked by the noted astronomer, Simon Newcomb, then head of the department of mathematics, to lecture on the theory of substitution groups and its applications to algebraic equations. In 1891 the lectures were published by invitation in the paper numbered 8 below, and in neighboring years Bolza wrote three further papers on the subject. One was a review of Cole's translation of Netto's *Theory of substitutions*, a book which aroused much interest in this country. The review was most thorough and scholarly, quite characteristic of Bolza, and much more comprehensive and critical than most reports of this sort.

Under the title "Miscellaneous" above I have grouped two papers on linear differential equations of the second order, Bolza's two publications concerning religious psychology, and other papers. But I wish to mention especially his autobiography *Aus meinem Leben* published privately in 1936. I had some time before urged him to write an account of his life and interests and was overjoyed when the result reached me. It is the source of much of the data in the sketch of his life presented here, and many readers besides his own students and colleagues will find great encouragement and interest in it.

By far the greatest of Bolza's interests was the calculus of variations, as indicated by the twenty-nine publications under III above. He was a student in the famous course of Weierstrass on the calculus of variations at the University of Berlin in 1879, but did not pursue the subject actively further until 1901 when he was invited to give a series of colloquium lectures at the summer meeting of the American Mathematical Society in Ithaca. He chose the calculus of variations as his subject. It was beautifully presented. The lectures much amplified, published in 1904 in the book numbered 33 in the bibliography

below, contained the contributions to that date, especially of Weierstrass, Kneser, and Hilbert, for problems of the calculus of variations in the plane in both non-parametric and parametric form, enriched by the able presentation of Bolza himself. This was only his beginning. He published in 1909 his *Vorlesungen über Variationsrechnung*, numbered 44 below, which is now a classic, indispensable to every scholar in the field, and much wider in its scope than his earlier book. The later volume contains besides the theory of problems in the plane, the theory of the problem of Lagrange with fixed end points, and an introduction to the problem of the calculus of variations for double integrals.

One of Bolza's important contributions was his extension and application of existence theorems for implicit functions and for solutions of differential equations. Before his time most theorems concerning implicit functions defined by equations of the form

$$f_i(x_1, \cdots, x_m; y_1, \cdots, y_n) = 0 \qquad (i = 1, \cdots, n)$$

were based upon the hypothesis that there is a single solution $(x; y) = (a; b)$ of these equations, and the purpose of the theorems was to show that for x in a neighborhood of a there exists a uniquely defined system of solutions

$$y_i(x_1, \cdots, x_m) \qquad (i = 1, \cdots, n)$$

near the point $y = b$. Bolza assumed a set P of solutions $(x; y) = (a; b)$, instead of a single one, and proved the existence of solutions defined over a neighborhood of the projection X of P in x-space. Similarly for a system of differential equations

$$f_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \qquad (i = 1, \cdots, n)$$

he assumed the existence of a solution arc $y_i(x)$ $(x_1 \leqq x \leqq x_2;$ $i = 1, \cdots, n)$ and proved that it could be imbedded in an n-parameter family of neighboring solutions. These theorems or their equivalents, as Bolza showed and as other students of the subject now well recognize, are essential in many instances to a rigorous theory of the calculus of variations.

Two of Bolza's papers, 53 and 56 in the bibliography below, have interested me especially. In them he presents a formulation of a new and very general problem of the calculus of variations which includes as special cases most of the problems which have been studied hitherto, and which has come to be known as the problem of Bolza.

The problem in parametric form is to find in a class of arcs $y_i(t)$ $(i=1, \cdots, n; t_0 \leqq t \leqq t_1)$ satisfying equations of the form

$$\phi_\alpha(y, y') = 0 \qquad (\alpha = 1, \cdots, p)$$

$$\psi_\beta(y) = 0 \qquad (\beta = 1, \cdots, q)$$

and end conditions of the form

$$\chi_\gamma[y(t_0), y(t_1)] = 0 \qquad (\gamma = 1, \cdots, r)$$

one which minimizes a sum

$$I = G[y(t_0), y(t_1)] + \int_{t_0}^{t_1} f(y, y')dt.$$

In these expressions the symbol y stands for the set $(y_1, \cdots, y_n)$, and similarly for y'. Bolza deduced first necessary conditions for a minimum and showed how the problems of Lagrange and Mayer are unified under his theory. He distinguishes with great clarity between so-called normal and abnormal minimizing arcs in cases much more general than had been considered before. In the second of the two papers he studies problems for which some of the side conditions written above are inequalities in place of equalities. It is impossible to describe or even mention here all of the contributions which Bolza made to the calculus of variations. He added liberally to the theory in the plane and to the problem of Lagrange, and improved them in many important respects. His methods are models which have greatly influenced many workers in the field.

The group of papers listed as IV above is a small one. Bolza was a great friend and admirer of E. H. Moore. The paper 46 is devoted to the formulation and proof of a theorem in *General analysis* which generalizes and unifies a multiplier rule of the calculus of variations and a theorem of elementary analytic geometry in the spirit so often formulated by Moore. In the summer of 1913 Bolza gave one of his inimitable lecture courses at the University of Chicago on integral equations. The lectures were written up by W. V. Lovitt who afterward published them with modifications in book form (see no. 54 of the bibliography). When Bolza returned to Germany after his summer in Chicago he had the strong feeling that Moore's *General analysis* and the accompanying theory of integral equations had been stated in print only very concisely, and that they should be much better known. He therefore published in the Jahresbericht an introduction of some fifty-six pages to Moore's theory, the paper num-

bered 55 below. It is written in Bolza's finest style and should be most helpful and suggestive to all students of the theory.

It would require a book to describe all of the research which Bolza published, even in the calculus of variations alone, and I shall content myself with the brief indications in the paragraphs which precede. His work was always meticulously precise, and was usually accompanied by a most scholarly and appreciative analysis of publications in the same neighborhood which preceded his own. Wherever his interest was aroused he has left valuable and indelible impressions in the field of mathematical research.

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The list is chronological.

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2. *Über die Reduction hyperelliptischer Integrale erster Ordnung und erster Gattung auf elliptische, insbesondere über die Reduction durch eine Transformation vierten Grades*, Dissertation, Göttingen, 1886, 39 pp.

3. *Über die Reduction hyperelliptischer Integrale erster Ordnung und erster Gattung auf elliptische durch eine Transformation vierten Grades*, Math. Ann. vol. 28 (1887) pp. 447–456; this is a review of the results in no. 2 above.

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G. A. BLISS

BOOK REVIEW

Vector and tensor analysis. By H. V. Craig. New York and London, McGraw-Hill, 1943. 14+434 pp. $3.50.

During the last decade, there has been considerable emphasis on the presentation of subjects in textbooks from the axiomatic viewpoint. This approach implies carefully worded definitions, axioms, and theorems. Further, in this point of view, the stress is placed on the analytical and logical rather than the geometrical and physical aspects of a subject. The type of presentation associated with this view has furnished interesting and valuable textbooks in such introductory subjects as college algebra and calculus. Previously, no such presentation had been attempted for a senior-graduate level text in vector and tensor analysis. Craig has admirably presented vector and tensor analysis in the light of this analytical-logical viewpoint.

In writing such a text on vector and tensor analysis, a subject whose origins and developments are closely connected with geometry and physics, an author faces many problems. Perhaps one of the most difficult of these problems is concerned with the author's treatment of differentials. From the logical viewpoint, differentials are non-essential tools since all their functions may be performed by derivatives. However, differentials act in two very important roles in vector and tensor analysis. First, the literature, both past and present, of the subject abounds in the use of differential notation. Second, differentials are useful to both the geometer and the physicist in interpreting his results. From these interpretations, many new results have been obtained, often by proofs which are incorrect from the modern point of view. However, rigor and concise thinking are fundamental to the author's approach to the subject. Hence, he has chosen to omit the treatment of differentials. Another important problem is the treatment of coordinate transformations and invariants. Very few introductory texts to vector and tensor analysis furnish an adequate treatment of this important topic. The principal difficulty is that a proper presentation of this topic requires that the student possess a knowledge of determinants and linear transformations. Because of the present author's analytical approach, he is able to offer an excellent account of this subject. In fact, the concept of invariance dominates the greater part of the book.

The book is divided into four parts; Part A deals with advanced calculus; Part B with elementary vector analysis; Part C with tensors and extensors; Part D with applications.

Part A is concerned with those parts of advanced calculus which are frequently used in the remainder of the text. Chapter I points out some common errors in reasoning and offers suggestions on reading to various types of students. The real number system, continuity, differentiability and integrability of one function of one real variable are carefully treated in the next chapter. In chapter III, the author considers the analogous properties of a function of two or more real variables. Implicit functions, the distinction between iterated and multiple integrals and the relation of these integrals are discussed. Geometry is introduced into the text in the next chapter via a discussion of the notion of parameterized arcs. In particular, Theorem (A 15.1) should be mentioned. This interesting result is neglected in most texts on advanced calculus. The chapter closes with a treatment of some topics in the calculus of variations. In particular, the author discusses the Euler equations in order to provide a base for a later consideration of the theory of geodesics. An excellent account of the ϵ-systems and δ-systems, determinants and coordinate transformations, brings Part A to a close. Although the group concept is completely developed in the treatment of coordinate transformations, unfortunately, the specific term "group" is nowhere defined nor used. The summation convention is introduced in Part A and the role of the Jacobian in integral transformations is clearly presented. Further, the problems in this part and the following parts of the book are well chosen and serve to supplement the theorems of the text proper.

Part B deals principally with the standard topics of vector analysis. The presentation differs considerably from that found in most texts in that: (1) definitions, axioms, theorems are clearly stated; (2) considerable stress is placed upon invariants. In chapter VI, the properties of vectors are defined, the scalar and vector products are introduced as invariants, and the triple scalar and vector products are discussed. It should be noted that with the aid of the ϵ-system, the author offers an easy proof of the usually difficult triple vector expansion. Chapter VII contains an axiomatic approach to n-dimensional vector spaces and linear manifolds. Following the modern approach to geometry, the author, next, introduces a metric in his n-space. An interesting exposition of the Schmidt orthogonalization process for positive definite and indefinite metrics concludes the chapter. Chapter VIII contains a discussion of the elements of the differential geometry of curves and surfaces, the gradient, divergence, and curl, and an introduction to tensor and extensor transformation theory. In concluding Part B, the author discusses the important integral transformations of vector analysis—the Gauss,

Stokes, and Green theorems. These topics—particularly Stokes' theorem—are well treated. This section contains only one troublesome typographical error: on page 159, the expression $(\gamma-\omega\times R)$ should read $\gamma-(\omega\times R)$. Further, the term "unitary orthogonal" on page 174 appears to be misapplied. In the literature, this term is reserved for vectors which are functions of a complex variable.

Part C deals with the theory of weighed tensors and extensors. Since the author has made important research contributions to extensor theory, his presentation of this subject should prove valuable to potential research workers in this field. Chapter X deals with the algebra of weighed tensors and extensors—the addition, multiplication, and contraction laws. The next chapter presents the differential calculus of tensors and extensors. Fundamental extensors, the fundamental tensors, and the Christoffel symbols are introduced in terms of derivatives of the radius vector in a Euclidean n-space. By a contraction of these fundamental extensors, the author obtains the ordinary covariant derivative. This procedure gives the combined theory of extensors and tensors considerable unity. A treatment of geodesics, geodesic coordinates, and the Riemann-Christoffel tensor concludes this part of the text. There is one minor point in notation which may cause some confusion. The author uses a pair of brackets around any two indices to indicate their antisymmetric nature. However, in a few instances, brackets are inserted about three indices without adequate explanations. Aside from this omission, part C is clearly written and constitutes a very good introduction to the theory of tensors and extensors.

Part D applies the material developed in parts B and C to classical dynamics and relativity. The first chapter in this section discusses such topics as moving coordinate systems, kinetic and potential energy of a particle and a system of particles, and the mechanics of continuous media. An interesting feature in the treatment of this last topic is the introduction of the Newtonian (mass constant) form and special relativity (mass variable) form of the equations of motion. Chapter XIII furnishes a discussion of the basis of special relativity, the Lorentz-Einstein transformation, applications of this transformation to velocity, acceleration, force, and mass, and finally a discussion of the energy-momentum tensor. The final chapter gives an introduction to General Relativity. In particular, the computation of the Schwarzchild line element is given in considerable detail.

This book presents vector and tensor analysis from an interesting point of view. It should provide a valuable addition to mathematical literature.

N. COBURN

NOTES

Lectures on matrices by J. H. M. Wedderburn, volume 17 of the American Mathematical Society Colloquium Publications, is again available. A limited number of copies have been prepared photographically from the original edition. This reprint has an expanded bibliography but otherwise contains exactly the material of the original edition. The price is unchanged, $3.35, with the usual 25 per cent discount to members of the Society.

The Association for Symbolic Logic has elected Dr. Alfred Tarski of the University of California as president and Professor C. A. Baylis of Brown University as vice president. Dr. C. G. Hempel of Queens College, Professor Karl Menger of the University of Notre Dame, and Professor Saunders MacLane of Harvard University have been elected members of the Executive Committee.

Dr. J. H. C. Whitehead of Balliol College, Oxford, has been elected a fellow of the Royal Society.

Professor Saunders MacLane of Harvard University has been elected a member of the American Academy of Arts and Sciences.

Professor G. D. Birkhoff and Harlow Shapley of Harvard University have been elected honorary members of the Mexican Mathematical Society. Professor Birkhoff has also been elected a corresponding member of the Mexican National Academy of Sciences.

Assistant Professor A. E. Taylor of the University of California at Los Angeles has been elected a corresponding member of the National Academy of Sciences of Lima.

Professor André Weil of Lehigh University has been awarded a Guggenheim fellowship. He will prepare a book on the foundations of algebraic geometry.

Dr. Warren Weaver of the Rockefeller Foundation, Professor E. P. Wigner of Princeton University, and Professor E. T. Whittaker of the University of Edinburgh have been elected members of the American Philosophical Society.

Lord Bertrand Russell has been elected a fellow of Trinity College, Cambridge.

Professor Alan Hazeltine of Stevens Institute of Technology has resigned.

Associate Professor B. W. Jones of Cornell University has been promoted to a professorship.

Associate Professor L. L. Lowenstein of Alfred University, Alfred, New York, has been promoted to a professorship.

Associate Professor L. F. Ollmann of Hofstra College, has been promoted to a professorship.

Assistant Professor G. L. Walker of the University of Delaware has been appointed to an acting assistant professorship at Cornell University.

Dr. C. P. Wells of Michigan State College has been promoted to an assistant professorship.

Mrs. A. W. Goodman has been appointed to an instructorship at Vassar College, and Dr. Murray Mannos to an instructorship at the University of Notre Dame.

Professor Emeritus A. E. Jolliffe of the University of London died March 17, 1944 at the age of seventy-three years.

Mr. Arne Fisher, mathematician with the Western Union Telegraph Company in New York, died April 8, 1944 at the age of fifty-seven years.

Professor H. C. Hicks of Carnegie Institute of Technology died April 14, 1944. He had been a member of the Society since 1923.

Dr. Clyde Wolfe of the Radiation Laboratory of the University of California died March 25, 1944.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

172. Warren Ambrose: *A variation of a theorem of I. E. Segal on the group ring of a compact group.*

It is shown that the L_2-ring of a compact group is a direct sum of full matrix algebras. This is done using the special properties of the L_2-ring, and its ideal theory, without reference to the classical theory of group representations, though the proof parallels, in part, the proof of the Peter-Weyl theorem. (Received April 14, 1944.)

173. Reinhold Baer: *Crossed isomorphisms.*

The 1:1 correspondence f, mapping the elements in the group G upon the elements in the group H, is termed a crossed isomorphism of G upon H, if there exists to every element x in G an endomorphism $e(x)$ of G such that $(u^{e(v)}v)^f = u^f v^f$ for u, v in G. The position of crossed isomorphisms between ordinary isomorphisms and 1:1 correspondences is discussed and their relation to projectivities of G upon H is investigated. If the endomorphism $e(x)$ of G maps every subgroup of G upon itself, then f is termed an integral crossed isomorphism. All the groups that are integral crossed isomorphic maps of abelian groups are determined. (Received April 28, 1944.)

174. R. J. Duffin and R. S. Pate: *Structure elements of quasigroups.* II.

This paper continues the study of the concepts of *rank* and *left associativity* (Duke Math. J. vol. 10 (1943) p. 743). A *left set* is a set L of elements such that $(Lx)y = L(xy)$ for every pair (x, y) of elements of Q. The *middle rank* of Q is the number of permutations of the columns of the multiplication table of Q which are simultaneously permutations of the rows. The ranks of Q divide its order. A Q of middle rank r contains a sub-Q of order r whose elements consist of all the middle associative elements $\{m\}$, that is, $x(my) = (xm)y$. These considerations furnish a practical test to determine whether a quasigroup is a group because the rank of a group is equal to the order. The meet, join and product of left sets are left sets. The set of elements $\{a\}$ such that $La = L$ is both a left set and a sub-Q. Unlike group theory, quotient Q's may be defined even when there are no sub-Q's. The lattice of quotient Q's is modular. The existence of a quotient quasigroup is equivalent to the existence of a set C satisfying certain associative and commutative laws. Permutation of rows, columns or symbols of the multiplication table leaves both rank and the lattice of weak quotients invariant. (Received May 31, 1944.)

175. R. J. Duffin and R. S. Pate: *Structure elements of quasigroups. III.*

This paper is a study of the three groups generated by the right and left substitutions of a quasigroup; G_r, G_l, and G. Group theoretic theorems concerning systems of imprimitivity are found to be useful in this connection. A quotient expansion of Q is equivalent to a system of imprimitivity of G. The lattice of quotients is isomorphic to a certain (synthetic) lattice of intransitive normal subgroups of G. If K is the subgroup corresponding to the quotient Q', then the G of Q' is the abstract group G/K. The Jordan-Hölder and Schrierer-Zassenhaus developments follow from these considerations. A left coset expansion of Q is equivalent to a system of imprimitivity in G_r. The lattice of left coset expansions is isomorphic to a certain lattice of subgroups of G_r. These results are related to abstract groups by employing a method of R. Baer (Trans. Amer. Math. Soc. vol. 46 (1939) pp. 110–141). Special types of quotient quasigroups are classified by the isotopies sending them into simpler types. A general feature noted is that the associative properties of the quasigroup are often reflected as commutative properties of the groups. (Received May 31, 1944.)

176. O. F. G. Schilling: *Automorphisms of fields of formal power series.*

Let $F=\Omega\{t\}$ be a field of power series of one variable over the field Ω. This note deals with the structure of the automorphism group of F/Ω. It is shown how the concepts of the Hilbert theory can be carried over so as to provide universal ramification groups for the totality of subfields $\Omega \subset L \subset F$ over which F is normal. (Received May 31, 1944.)

177. O. F. G. Schilling: *Noncommutative valuations.*

Let D be a division algebra and V a valuation on D with values in a simply ordered 1-group. It is shown that the ideal theory in D with respect to V can be developed as in the commutative case provided proper safeguards are observed. It turns out that the division algebras in question must be transcendental over their centers. Certain algebras may be considered as crossed extensions of division algebras by 1-groups. (Received May 31, 1944.)

178. S. M. Ulam and C. J. Everett: *On ordered groups.*

An o-group G is an l-group (G. Birkhoff, *Lattice ordered groups*, Ann. of Math. vol. 43 (1942)) with the lattice property weakened to that of Moore-Smith. An o-group is embeddable in a complete ordered group if and only if it is integrally closed. If the commutator group of an integrally closed o-group is in its center, it is commutative. The conjecture of the "suggestion" on p. 329, loc. cit., is thus proved. Counterexamples are given for Problems 1, 2, loc. cit. The group of functions $ax+A$, a, A real, $a>0$, under composition admits no integrally closed order. Every l-group is embeddable in a sequence complete group in the sense of o-convergence. The results of Everett, *Sequence completion of lattice moduls*, Duke Math. J. vol. 11 (1944), are extended to noncommutative groups. Various orders in the free group of two generators establish curious properties of the group of topological transformations of the line into itself. (Received April 25, 1944.)

ANALYSIS

179. R. P. Agnew: *A family of bounded sequences summable M.*

A simple example is given of a family of bounded sequences, summable M, which is not a separable subset of the space B of bounded sequences. (Received May 2, 1944.)

180. R. P. Agnew: *Convergence fields of methods of summability.*

Let A be a multiplicative matrix method of summatibility with multiplier ρ. If $\rho=0$, each sequence x_n which oscillates sufficiently slowly is summable A to 0. If $\rho\neq 0$ and x_n is summable A to L, then the sequence $(x_n-L/\rho)\xi_n$ is summable A to 0 if the sequence ξ_n oscillates sufficiently slowly. (Received May 16, 1944.)

181. I. S. Reed: *On the solution of a general transform.*

The purpose of this paper is to give a brief extension to the solution of a Watson transform with an unsymmetrical kernel. Use has been made of the work that has been done by Hardy, Watson, Titchmarsh, Goodspeed, and others. (Received April 26, 1944.)

182. Tibor Rado: *On the Geöcze area of Fréchet surfaces.* Preliminary report.

The Geöcze area of a surface S is defined in terms of projections upon the coordinate planes. Perfecting previous theories originating with the work of Geöcze, Reichelderfer (Trans. Amer. Math. Soc. vol. 53 (1943) pp. 251–291) introduced and studied an *essential area.* In his definition (loc. cit. p. 274) we replace *simple Jordan regions* by *domains of any connectivity,* and obtain an area to be denoted by $G(S)$, where S stands for a Fréchet surface of the type of the sphere or the disc. In this paper we develop a comprehensive theory of $G(S)$ as a foundation for the theory of the area. One of the results states that if the Lebesgue area $L(S)$ is finite, then $G(S)=L(S)$. Essential use is made of methods and results developed in recent years by Reichelderfer, Morrey, Youngs, and the author. (Received April 19, 1944.)

183. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* VI.

The author constructs equations in iterative compositions satisfied by specific functions of variables and derived from conditions of invariance of iterative compositions of these functions under substitution groups on the variables of the latter. For example, functional equations satisfied by arbitrary functions of the type $\gamma(x_1+x_2+\cdots+x_n)$ are obtained by assuming that the function $\beta\{\alpha(x_1+x_2+\cdots+x_m)+\alpha(y_1+y_2+\cdots+y_m)+\cdots+\alpha(t_1+t_2+\cdots+t_m)\}$ corresponds to a set of imprimitive systems of a given imprimitive substitution group G_S on the variables $x_i, y_i, \cdots, t_i$ $(i=1, 2, \cdots, m)$. These equations express invariance (in the sense indicated) under G_S of the corresponding functional composition $\phi\{f(x_1, x_2, \cdots, x_m), f(y_1, y_2, \cdots, y_m), \cdots, f(t_1, t_2, \cdots, t_m)\}$. A set of functional equations of the preceding type corresponds to any abstract group G with subgroup H, since G can be represented as a (regular) substitution group G_S simply isomorphic to G and such that G_S is imprimitive with set of imprimitive systems corresponding to H. Functional equations are also constructed when the arbitrary function β has as arguments functions whose variables are not necessarily equal in number. (Received May 29, 1944.)

184. A. R. Schweitzer: *On functional equations representing abstract groups.*

Let $f(x_1, x_2, \cdots, x_{n+1}) = a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1}$ and assume $a_i \cdot (a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1}) = (a_ia_1)x_1 + (a_ia_2)x_2 + \cdots + (a_ia_{n+1})x_{n+1}$; then the equation $f(u_1, u_2, \cdots, u_{n+1}) = f(L_1, L_2, \cdots, L_{n+1})$, where $u_i = f(x_{i1}, x_{i2}, \cdots, x_{in+1})$ and L_i are certain formal sums of the x's, represents a class of multiplication tables for the a's. Group representation by a single functional equation follows if the associated multiplication table defines an abstract group G (with a_1 as the identical element). In this case the formal sums $L_1, L_2, \cdots, L_{n+1}$ are, in effect, derived from L_1 by applying to the variables of L_1 substitutions of a regular group simply isomorphic to G. The preceding representation is of formal additive, that is, special type. Group representation of general type is obtained by subjecting the variables x_{ij} in the composition $f(u_1, u_2, \cdots, u_{n+1})$ to conditions of invariance of the latter suggested by the invariance of the formal sums serving as arguments of the function $f(L_1, L_2, \cdots, L_{n+1})$. An analogous representation of linear algebras is discussed. (Received May 29, 1944.)

185. A. R. Schweitzer: *Functional relations valid in the domains of abstract groups and Grassmann's space analysis.*

The function of group elements $f(x_1, x_2, \cdots, x_{n+1}) = x_1 \cdot x_2^{-1} \cdot x_3, \cdots, x_{n+1}^{+1}$ when n is odd satisfies the author's postulates generalizing the group concept to $(n+1)$-ary composition (abstract 50-3-73). When n is even, the preceding function satisfies the postulates: 1. $f(u_1, u_2, \cdots, u_{n+1}) = f(u, t_1, t_2, \cdots, t_n)$ where $u_i = f(x_i, t_1, t_2, \cdots, t_n)$ and $u = f(x_1, x_2, \cdots, x_{n+1})$. 2. Given $f(x_1, x_2, \cdots, x_{n+1})$ there exists $\phi(x_1, x_2, \cdots, x_{n+1})$ such that $f\{\phi(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\} = x$ and $\phi\{f(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\} = x$. 3. $f(x, x, \cdots, x) = x$. 4. The set S of elements x_i is closed under the compositions f and ϕ. When $n = 1$ the postulates are satisfied by $f(x_1, x_2) = x_2 \cdot x_1 \cdot x_2^{-1}$ and $f(x_1, x_2) = x_2 \cdot x_1^{-1} \cdot x_2$. A solution containing an arbitrary function is $f(x_1, x_2, \cdots, x_{n+1}) = x_1 \cdot x_{n+1}^{-1} \cdot \alpha(w_2, w_3, \cdots, w_n) \cdot x_{n+1}$ where $w_i = x_i \cdot x_{n+1}^{-1}$. In Grassmann's space the postulates are satisfied if $f(x_1, x_2, \cdots, x_{n+1}) = a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1}$ where the x_i are points and the a_i belong to a field with sum equal to unity. In the above postulates the functions f and ϕ are dual to one another; if $f\{f(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\} = x$ then $f = \phi$. Also f is distributive over ϕ. For $n = 1$ the restricted associative properties, $f\{x_1, f(x_2, x_1)\} = f\{f(x_1, x_2), x_1\}$ and $f\{x_1, \phi(x_2, x_1)\} = \phi\{f(x_1, x_2), x_1\}$ hold. (Received May 29, 1944.)

186. W. J. Thron: *A family of simple convergence regions for continued fractions.*

In this paper a new two parameter family of simple convergence regions for continued fractions (1) $1 + K(-c_n^2/1)$ is obtained. It is further shown that the regions cannot be improved except possibly by the removal of an epsilon. A simpler though less inclusive convergence criterion is provided by the following corollary of the main theorem: the continued fraction (1) converges if for all $n \geqq 1$ its elements $c_n = re^{i\theta}$ satisfy the conditions (a) $r \leqq (d - \epsilon + 1/2) \cos \alpha \sec(\theta - \alpha)$, $-\pi/2 + \alpha < \theta \leqq \beta_1$, (b) $r \leqq (d - 1/2) \cos \alpha \sec(\theta - \alpha)$, $0 \leqq \theta < \pi/2 + \alpha$, (c) $r = 0$ otherwise, (d) $\sum |b_n| = \infty$, where $b_1 = 1/c_1^2$, $b_n = 1/c_n^2 b_{n-1}$. Here $-\pi/2 < \alpha < \pi/2$, $0 \leqq d \leqq 1/2$, $0 < \epsilon < d/2$ and $\cot \beta_1 = -(\tan \alpha + ((d + 1/2)/d)^{1/2} \sec \alpha)$. For $d = 0$ this result reduces to the well known parabola theorem, an isolated result of Paydon (abstract 47-11-473) is also included in this theorem. (Received June 1, 1944.)

APPLIED MATHEMATICS

187. A. N. Lowan and H. E. Salzer: *Formulas for complex interpolation.*

Whenever an analytic function is tabulated for arguments z_ν located at equidistant points along any straight line in the z-plane, the application of the Lagrange-Hermite interpolation formula, which approximates the function by a complex polynomial of degree n passing through any $n+1$ points, leads to the sum from $\nu=-[n/2]$ to $\nu=[(n+1)/2]$ of $f(z)\sim\sum A_\nu^{(n)}(p)f(z_\nu)$ where $A_\nu^{(n)}(P)$ are polynomials in $P=(z-z_0)/h$ and h is the complex tabular interval and where $[x]$ stands for the largest integer in x. Since in general z is not on the straight line containing the point z_ν, P is complex ($=p+iq$). Expressions for the real and imaginary parts of $A_\nu^{(n)}(P)$ as functions of p and q were obtained for the cases of three, four, five and six point interpolation and arranged in a form especially suited for computational purposes. These expressions are particularly applicable to the cases of functions tabulated along rays through the origin, as for instance: the table of the Bessel functions $J_0(z)$ and $J_1(z)$ computed by the Mathematical Tables Project (Columbia Press, 1943), the forthcoming table of $Y_0(z)$ and $Y_1(z)$, H. T. Davis' table of $1/\Gamma(z)$ and Kennelly's tables of complex circular and hyperbolic functions for Cartesian and polar arguments. (Received April 12, 1944.)

188. H. E. Salzer: *Coefficients for mid-interval numerical integration with central differences.*

Coefficients in the formula for numerical integration from mid-interval to mid-interval using central differences were calculated to the extent where one can employ central differences up to the forty-ninth order. (Previous calculations have gone only as far as coefficients of the seventh central difference.) The coefficients K_{2s} occur in the following formula, which is also known as the first Gaussian summation-formula: $(1/h)\int_{a-h/2}^{a+(n-1/2)h} f(x)dx=[f(a)+f(a+h)+\cdots+f(a+(n-2)h)+f(a+(n-1)h)]+\sum_{s=1}^{m-1}K_{2s}[\delta^{2s-1}f(a+(n-1/2)h)-\delta^{2s-1}f(a-h/2)]+nh^{2m}K_{2m}f^{(2m)}(\xi)$. Due to the extreme rapidity with which the coefficients K_{2s} decrease with increasing s, the calculated table can be used with high accuracy in either integration or summation even though successive differences might not show the slightest tendency to decrease for the given interval. The quantities K_2 to K_{20} are given exactly and K_{22} to K_{50} are given to 18 decimals, accurate to within 0.6 units in the 18th place. These coefficients were checked by two cumulative recursion formulas, by differencing of the ratios K_{2s}/K_{2s+2}, and by a numerical example. (Received May 13, 1944.)

189. H. E. Salzer: *Table of coefficients for differences in terms of the derivatives.*

A table was prepared which lists the exact values of the coefficients $B_{m,s}$ for $m=1, 2, \cdots, 20$ and $s=m, \cdots, 20$ in the formula of Markoff which expresses the mth advancing difference in terms of the derivatives according to the equation $\Delta_h^m f(a)=\sum_{s=m}^{n-1}B_{m,s}h^sD^sf(a)+B_{m,n}h^nD^nf(\eta)$ where h denotes the tabular interval. The quantity $B_{m,s}$ in Milne-Thomson's notation is equal to the $(s-m)$th Bernoulli number of order $-m$, divided by $(s-m)!$ and in Jordan's notation $B_{m,s}$ is equal to $m!/s!$ multiplied by $\mathfrak{S}_s^m$ where $\mathfrak{S}_s^m$ is the Stirling number of the second kind. The coefficients $B_{m,s}$ were calculated by first obtaining the Stirling numbers $\mathfrak{S}_s^m$ (using their well

known recursion relationships, and checking by a neat formula given in Jordan's *Calculus of finite differences*) and then multiplying by $m!/s!$. The numbers $B_{m,s}$ are expressed in lowest terms. (Received April 20, 1944.)

190. Seymour Sherman, J. DiPaola, and H. F. Frissel: *Routh's discrimant, flutter, and ground resonance.* Preliminary report.

Routh's criterion for the stability of the solutions of system of linear differential equations with constant coefficients is extended to cover cases arising in airplane flutter and helicopter ground resonance calculations. With this new tool, the stability of the flutter "polynomial" at a given reduced frequency for more than two degrees of freedom can be determined in one-fifth of the time hitherto required. (Received May 29, 1944.)

GEOMETRY

191. Edward Kasner and John DeCicco: *Isothermal families in general curvilinear coordinates, and loxodromes.*

If the square of the linear element of a surface Σ is given in isothermal coordinates (x, y) by $ds^2=E(x, y)(dx^2+dy^2)$, then the family of curves $g(x, y)=$const. on Σ is isothermal if and only if $(\partial^2/\partial x^2+\partial^2/\partial y^2)$ arc tan $g_y/g_x=0$. In the present paper, the authors obtain the necessary and sufficient condition that $g(x, y)=$const. represent an isothermal family when (x, y) are general curvilinear coordinates. This gives a large extension of Lie's theorem. The condition is simpler when the parametric curves form an orthogonal net. As an application, the condition is obtained that $g(x, y)=$const. represent an isothermal family upon the Cartesian surface $z=f(x, y)$. Finally the condition is found that the level curves of the surface be an isothermal family. This is applied to the mapping of loxodromes, showing that they can be represented by straight lines for a sphere (Mercator) and spheroid (Lambert), but not for an ellipsoid of three unequal axes. Use is made of Kasner's theorem in Math. Ann. (1904). (Received April 20, 1944.)

192. Abraham Seidenberg: *Valuation ideals in polynomial rings.*

A constructive study of the valuation ideals in a polynomial ring $\mathfrak{O}=K[x, y]$ in two indeterminates, where K is an algebraically closed (ground-) field, is made. Let $\mathfrak{q}_1, \mathfrak{q}_2, \cdots$ be the Jordan sequence of v-ideals belonging to a valuation B of Σ/K, where Σ is the quotient field of $\mathfrak{O}$, and let $\mathfrak{q}_{i_j}$ be the jth ideal such that $v(\mathfrak{q}_{i_j})$ is not in the additive group generated by $v(\mathfrak{q}_1), \cdots, v(\mathfrak{q}_{i_{j-1}})$. A tool corresponding to the Puiseux series expansion for a valuation, which is available if K is of characteristic 0 but not in general, is found in introducing certain polynomials f_{i_j} such that $v(f_{i_j})=v(\mathfrak{q}_{i_j})$. Considerations are reduced to valuations of rational rank 2. If B is of rational rank 2, place $v(\mathfrak{q}_1)=1$ and let τ be the least irrational value assumed by elements of $\mathfrak{O}$. The description of the v-ideals in $\mathfrak{O}$ for B is intimately connected with the approximants and quasiapproximants to a certain integral multiple of τ. In particular, a simple 0-dimensional v-ideal $\mathfrak{q}_i$ is characterized in terms of the values of $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$. This characterization yields a proof that the transform of a simple v-ideal under a quadratic transformation is simple. If $\mathfrak{q}_i$ is not simple, an explicit factorization of $\mathfrak{q}_i$ in terms of the mentioned values is given. (Received May 22, 1944.)

193. A. H. WHEELER: *One-sided polyhedra from the five regular solids.*

If planes are passed through a polyhedron in such a way as to permit removing parts and leaving cells which are joined along their edges, and adjoining cells are bounded by one or more planes which are common to the two cells, then one-sided polyhedra can be derived. In this paper the method is applied to the five regular solids and several forms are shown constructed of plastic material. The intersecting planes may be so disposed as to cut away the vertices, the edges in whole or in part, and parts of the faces of the original solids. In particular, in the case of the regular icosahedron, a part of a regular dodecahedron of the second species may be removed from the interior of the solid leaving a group of cells distributed along the edges of the original solid, and the cells can be entered or traversed in more than one way. Two types of one-sided octahedra are shown. (Received May 26, 1944.)

194. Oscar Zariski: *The theorem of Bertini concerning the variable singular points of a linear system of varieties.*

If the ground field k of an algebraic irreducible r-dimensional variety V/k is extended by the adjunction of indeterminates $u_1, u_2, \cdots, u_m$, we denote by V/K the extended variety over $K=k(u)$ which has the same general point (ξ) as V/k. Each subvariety W/k of V/k has similarly a unique extension W/K on V/K, and each subvariety W^*/K of V/K has a unique contraction W/k on V/k. Given $m+1$ linearly independent polynomials $f_i(\xi)$, $i=0, 1, \cdots, m$, there is a unique $(r-1)$-dimensional irreducible subvariety F^*/K of V/K whose general point (η) satisfies the equation $f_0(\eta)+u_1f_1(\eta)+\cdots+u_mf_m(\eta)=0$. The main result is as follows: (1) W/k is a base variety of the linear system $|F|$ defined on V/k by $f_0+\lambda_1f_1+\cdots+\lambda_mf_m=0$, if and only if W/K is on F^*/K; (2) if W^*/K is a singular subvariety of F^*/K, then the contraction W/k is either singular for V/k or is a base variety of $|F|$. The theorem of Bertini *in its classical formulation* ("the variable singular points of $|F|$ are either singular for V/k, or lie on the base locus of $|F|$") is equivalent to (2), if k is of characteristic zero, and is false if k is of characteristic $p\neq 0$. (Received April 6, 1944.)

Topology

195. W. H. Gottschalk: *Orbit-closure decompositions and almost periodic properties.*

Let X be a metric space with metric ρ, let $f(X)\subset X$ be a continuous mapping, and let $h(X)=X$ be a homeomorphism. For $x\in X$, the set $\sum_{n=0}^{+\infty} f^n(x)$ is called the *semi-orbit* of x under f and the set $n\sum_{-\infty}^{+\infty} h^n(x)$ is called the *orbit* of x under h. The mapping f is said to be *pointwise almost periodic* provided that if $x\in X$, then to each $\epsilon>0$ there corresponds a positive integer N with the property that in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x))<\epsilon$. The mapping f is said to be *uniformly pointwise almost periodic* provided that to each $\epsilon>0$ there corresponds a positive integer N such that if $x\in X$, then in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x))<\epsilon$. The following theorems are proved: In order that the mapping f (homeomorphism h) give a semi-orbit-closure (orbit-closure) decomposition of X it is sufficient that f (h) be pointwise almost periodic; and in case X is locally compact (compact), this condition is also necessary. In order that the mapping f (homeomorphism h) give a continuous semi-orbit-closure (orbit-closure) decomposition of X it is sufficient that f (h) be uniformly pointwise almost periodic; and in case X is compact, this condition is also necessary. (Received May 15, 1944.)

196. Paul A. White: *On additive properties of compact sets.*

A property p^i concerned with i-dimensional Vietoris cycles of a compact set is called *additive* with respect to i if the following theorem is true. If the compact sets A and B have p^i and $A \cdot B$ has p^{i-1}, then $A \cdot B$ has p^i. It is shown that local i-connectedness and simple i-connectedness are two additive properties. Finally it is shown that the property $P(n, j)$ of being a compact, simply i-connected ($i=0, 1, \cdots, m-2$) lc^{n-2} embedded in E_n, having only simply-j-connected ($j \leq n-2$) generalized closed manifolds as boundaries of its complementary domains, is additive with respect to n and j simultaneously. (Received April 10, 1944.)

197. G. T. Whyburn: *Interior mappings into the circle.*

In this paper it is shown that any continuous transformation of a locally connected cyclic continuum M into the circle S is homotopic to an interior mapping of M into S. If M is not assumed to be cyclic, it is shown that any mapping $f(M)=S$ is homotopic to a mapping $\phi r(M)=S$ where $r(M)=M'$ is the monotone retraction of M onto the smallest A-set M' in M containing all cyclic elements on which f is essential and where $\phi(M')=S$ is interior. (Received April 14, 1944.)

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